

MATH 544: METHODS OF APPLIED MATHEMATICS II

Second Midterm Exam (SOLUTION)

April 25, 2016; 10.40-12.30

**QUESTIONS:** Choose any three of the following four problems

[35] 1. Suppose that  $k > 0$  is a constant and  $\nabla^2 u - k^2 u = 0$ . Formulate Dirichlet, Neumann, and Robin problems of this operator  $L = \nabla^2 - k^2$  and prove that each problem has at most one solution.

[35] 2. Let  $u \in C^2(D) \cap C^1(\bar{D})$ , where  $D$  is a bounded domain. Prove the following: (for solution see Lecture Notes 6, page 13)

(i) If  $\nabla^2 u \leq 0$  in  $D$  and  $u \geq 0$  on  $B$ , then  $u \geq 0$  in  $D$ ,

(Here  $B$  is the boundary of  $D$ )

ii) If  $\nabla^2 u \geq 0$  in  $D$  and  $u \leq 0$  on  $B$  then  $u \leq 0$  in  $D$ .

[35] 3. In two dimensions consider

$$\nabla^2 u = 0, \quad 0 < x < a, \quad 0 < y < b, \quad (1)$$

$$u(x, 0) = f(x), \quad u(x, b) = 0, \quad 0 \leq x \leq a, \quad (2)$$

$$u(0, y) = u(a, y) = 0, \quad 0 \leq y \leq b. \quad (3)$$

(a). Find the formal solution,

(b). Find reasonable restrictions on  $f(x)$  so that the formal solution is a solution.

[35] 4. (a) Solve the following eigen-value equation

$$y(x) = \lambda \int_0^{2\pi} \cos(x+s) y(s) ds.$$

(b) Obtain the most general solution of the integral equation

$$y(x) = f(x) + \lambda \int_0^{2\pi} \cos(x+s) y(s) ds.$$

when  $f(x) = x$  and  $f(x) = 1$ . Discuss all cases.

# Solution of mid term 2 problems

①  $L = \nabla^2 - k^2$  with  $k > 0$  and  $D$  is a closed and bounded domain with boundary  $B$

a) Dirichlet's problem

$$Lu = 0 \quad \text{in } D$$

$$u|_B = f$$

uniqueness:

$$\iint_D u \nabla^2 u \, dv = \iint_D [\nabla(u \nabla u) - \nabla u \cdot \nabla u] \, dv$$

$$\Rightarrow \iint_D (u \nabla^2 u + \nabla u \cdot \nabla u) \, dv = \iint_D \nabla(u \nabla u) \, dv$$

$$= \oint_B u \nabla u \, d\vec{S} = \oint_B u \frac{\partial u}{\partial n} \, dA$$

finally we have

$$\iint_D (u \nabla^2 u + \nabla u \cdot \nabla u) \, dv = \oint_B u \frac{\partial u}{\partial n} \, dA$$

Dirichlet's problem:  $\nabla^2 u = k^2 u$  in  $D$

Assuming that there exists two different solutions  $u_1$  and  $u_2$  of the <sup>same</sup> Dirichlet's problem. Define the difference function  $w = u_1 - u_2$ . This function satisfies

$$\nabla^2 w = k^2 w \quad \text{in } D$$

$$w = 0 \quad \text{on } B$$

Hence we obtain for  $w$

$$\iint_D (w \nabla^2 w + \nabla w \cdot \nabla w) dV = \oint_B w \frac{\partial w}{\partial n} dA = 0$$

$$\text{or } \iint_D [\nabla w \cdot \nabla w + k^2 w^2] dV = 0$$

hence,  $\nabla w = 0$ ,  $w = 0$  in  $D$

but this leads to  $u_1 = u_2$ . Contradiction to the assumption.

h) Neumann Problem:

$$Lu = 0 \quad \text{in } D$$

$$\left. \frac{\partial u}{\partial n} \right|_B = g(x)$$

Uniqueness: Following the steps in part (a) we arrive at again

$$\nabla w = 0, \quad w = 0$$

again this leads  $u_1 = u_2$  everywhere in  $D$  and on  $B$  (due to the continuity)

3) Robin problem

$$Lu = 0 \quad \text{in } D$$

$$u(x) + \alpha \frac{\partial u}{\partial n}(x) = h(x) \quad \text{on } B, \quad \alpha > 0$$

uniqueness: Following the same steps in part (a) the difference function  $w$  satisfies:

$$\iint_D (w \nabla^2 w + \nabla w \cdot \nabla w) dV = \oint_B w \frac{\partial w}{\partial n} dA$$

$$\nabla^2 w = 0 \quad \text{in } D$$

$$w(x) + \alpha \frac{\partial w}{\partial n} = 0$$

$$\Rightarrow \iint_D [k^2 w^2 + |\nabla w|^2] dV + \oint_B \alpha \left( \frac{\partial w}{\partial n} \right)^2 dA = 0$$

hence  $w = 0, \quad \nabla w = 0 \quad \text{in } D$

$$\frac{\partial w}{\partial n} = 0 \quad \text{on } B$$

by continuity  $w = 0$  everywhere  $D \cup B$   
but this is a contradiction. -

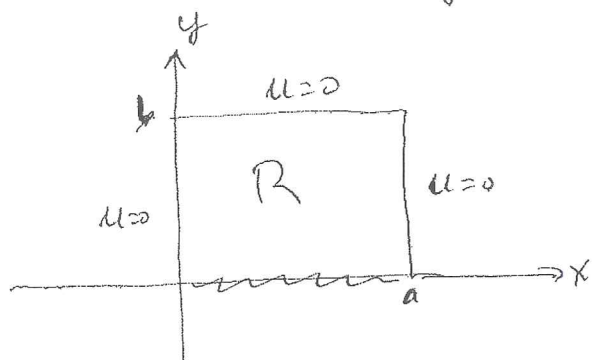
problem 3.

$$\nabla^2 u = 0, \quad 0 < x < a, \quad 0 < y < b$$

3.1

$$u(x, 0) = f(x), \quad u(x, b) = 0 \quad 0 \leq x \leq a.$$

$$u(0, y) = u(a, y) = 0, \quad 0 \leq y \leq b.$$



Soln:  $u(x, y) = \sum_{n=1}^{\infty} a_n \sinh \lambda_n (y-b) \cdot \sin(\lambda_n x)$  (1)

( $\lambda_n = \frac{n\pi}{a}, n=1, 2, \dots$ )

$$u(x, 0) = f(x) = - \sum_{n=1}^{\infty} a_n \sinh(\lambda_n b) \sin \lambda_n x$$
 (2)

$$\Rightarrow a_n \sinh \lambda_n b = - \frac{2}{a} \int_0^a \sin(\lambda_n x) f(x) dx$$

Justification: (1) series corresponding to  $u_{xx}$  or  $u_{yy}$  is

$$- \sum_{n=1}^{\infty} a_n \lambda_n^2 \sinh \lambda_n (y-b) \sin(\lambda_n x)$$

which should be

uniformly convergent

$$\left| \sum a_n \lambda_n^2 \sinh \lambda_n (y-b) \sin \lambda_n x \right| \leq \sum |a_n| \lambda_n^2 |\sinh \lambda_n (y-b)|$$

~~$$\leq \sum |a_n| \lambda_n^2 |\sinh \lambda_n (y-b)|$$~~

$$\leq \frac{2}{a} \sum \lambda_n^2 \left| \frac{\sinh \lambda_n (y-b)}{\sinh \lambda_n b} \right| \int_0^a |f(x)| dx$$

$$\frac{\sinh \lambda_n(y-b)}{\sinh \lambda_n b} = \frac{e^{\lambda_n(y-b)} - e^{-\lambda_n(y-b)}}{e^{\lambda_n b} - e^{-\lambda_n b}}$$

$$= \frac{e^{\lambda_n y - 2\lambda_n b} - e^{\lambda_n y}}{1 - e^{-2\lambda_n b}}$$

$$1 + e^{-2\lambda_n b} = 2$$

$$e^{-2\lambda_n b} \leq e^{-2\lambda_1 b}$$

n2,1

$$-e^{-2\lambda_n b} \leq -e^{-2\lambda_1 b}$$

$$1 - e^{-2\lambda_n b} \leq 1 - e^{-2\lambda_1 b}$$

$$\frac{1}{1 - e^{-2\lambda_n b}} \leq \frac{1}{1 - e^{-2\lambda_1 b}}$$

$$\Rightarrow \frac{\sinh \lambda_n(y-b)}{\sinh \lambda_n b} \leq \frac{e^{\lambda_n y - 2\lambda_n b} - e^{\lambda_n y}}{1 - e^{-2\lambda_1 b}}$$

We showed that

$$\begin{aligned}
 \left| \frac{\sinh \lambda_n (y-b)}{\sinh \lambda_n b} \right| &\leq \frac{1}{1-e^{-2\lambda_n b}} \left| e^{\lambda_n y - 2\lambda_n b} - e^{-\lambda_n y} \right| \\
 &\leq \frac{1}{1-e^{-2\lambda_n b}} \left| e^{\lambda_n y - 2\lambda_n b} - e^{-\lambda_n b} \right| \\
 &\leq \frac{1}{1-e^{-2\lambda_n b}} e^{-\lambda_n b} \left| e^{\lambda_n (y-b)} - 1 \right| \\
 &\leq \frac{2}{1-e^{-2\lambda_n b}} e^{-\lambda_n b} \quad \forall y \in (a, b).
 \end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
 \left| \sum a_n \lambda_n^2 \sinh \lambda_n (y-b) \sin \lambda_n x \right| &\leq \frac{4}{a(1-e^{-2\lambda_1 b})} \sum_{n=1}^{\infty} \lambda_n^2 e^{-\lambda_n b} \\
 &\leq \frac{4}{a(1-e^{-2\lambda_1 b})} \left( \int_0^a |f(x)| dx \right) \\
 &\quad \underbrace{\sum \lambda_n^2 e^{-\lambda_n b}}_{\text{convergent.}}
 \end{aligned}$$

Hence the series corresponding to the partial derivatives  $u_{xx}, u_{yy}$  are uniformly convergent if  $|f(x)|$  is integrable in  $(0, a)$ .

Justification of ②:  $u(x,0) = f(x) = -\sum a_n \sinh(\lambda_n b) \sin(\lambda_n x)$

with

$$a_n \sinh \lambda_n b = -\frac{2}{a} \int_0^a \sin(\lambda_n x) f(x) dx$$

•  $f(0) = f(a) = 0$

$f''$  is continuous in  $(0,a)$  then

the above series is uniformly convergent.

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$$\begin{aligned} a) \quad y(x) &= \lambda \int_0^{2\pi} \cos(x+s) y(s) ds \\ &= A \cos x - B \sin x \end{aligned}$$

where

$$\begin{aligned} A &= \lambda \int_0^{2\pi} \cos s y(s) ds \\ &= \lambda \int_0^{2\pi} \cos s (A \cos s - B \sin s) ds \\ &= \lambda A \pi \Rightarrow \lambda_1 = \frac{1}{\pi} \end{aligned}$$

$$\begin{aligned} B &= \lambda \int_0^{2\pi} \sin s y(s) ds \\ &= \lambda \int_0^{2\pi} \sin s (A \cos s - B \sin s) ds \\ &= -\lambda B \pi \Rightarrow \lambda_2 = -\frac{1}{\pi} \end{aligned}$$

Hence we have.

$$\lambda_1 = \frac{1}{\pi}, \quad y_1(x) = \cos x$$

$$\lambda_2 = -\frac{1}{\pi}, \quad y_2(x) = \sin x$$

$$\begin{aligned} b) \quad y(x) &= f(x) + \lambda A \cos x - \lambda B \sin x \\ \alpha &= \int_0^{2\pi} \cos s y(s) ds = \int_0^{2\pi} \cos s \left[ f(s) + \lambda A \cos s - \lambda B \sin s \right] ds \\ &= \beta_1 + \lambda \alpha \cdot \pi \Rightarrow \alpha (1 - \lambda \pi) = \beta_1 \end{aligned}$$

$$\beta = \int_0^{2\pi} \sin s y(s) ds = \int_0^{2\pi} \sin s \left[ f(x) + \lambda \alpha \cos s - \lambda \beta \sin s \right] ds$$

$$= \beta_2 - \lambda \beta \pi \Rightarrow \beta (1 + \lambda \pi) = \beta_2$$

i) if  $\lambda \neq \frac{1}{\pi} \text{ or } -\frac{1}{\pi}$

$$\Rightarrow y(x) = f(x) + \frac{\beta_1 \lambda}{1 - \lambda \alpha} \cos x - \frac{\beta_2 \lambda}{1 + \lambda \pi} \sin x$$

where  $\beta_1 = \int_0^{2\pi} \cos x f(x) dx$ ,  $\beta_2 = \int_0^{2\pi} \sin x f(x) dx$

ii) if  $\lambda = \frac{1}{\pi} \Rightarrow \beta_1 = \int_0^{2\pi} \cos x f(x) dx = 0$

$$y(x) = f(x) + \lambda \alpha \cos x - \frac{\beta_2 \lambda}{2} \sin x$$

$$= f(x) + \frac{\alpha}{\pi} \cos x - \frac{\beta_2}{2\pi} \sin x$$

$\alpha$  is arbitrary,  $\beta_1 = \int_0^{2\pi} \cos x f(x) dx = 0$

either there are infinitely many solutions if  $\beta_2 = 0$   
or there exist no solutions

iii) if  $\lambda = -\frac{1}{\pi} \Rightarrow \beta_2 = \int_0^{2\pi} \sin x f(x) dx = 0$

$$y(x) = f(x) + \frac{\lambda \beta_1 \cos x}{1 - \lambda \alpha} + \frac{1}{\pi} \beta \sin x$$

$$= f(x) + \frac{\beta_1}{2\pi} \cos x + \frac{\beta}{\pi} \sin x, \quad \beta \text{ is arbit}$$

$$\beta_2 = \int_0^{2\pi} \sin x f(x) dx = 0$$

c) when  $f(x) = x$

$$B_1 = \int_0^{2\pi} x \cos x \, dx = x \sin x \Big|_0^{2\pi} - \int_0^{2\pi} \sin x \, dx$$

$$= 0 + \cos x \Big|_0^{2\pi} = 0$$

$$B_2 = \int_0^{2\pi} x \sin x \, dx = -x \cos x \Big|_0^{2\pi} + \int_0^{2\pi} \cos x \, dx$$

$$= -2\pi$$

$$y(x) = x + \frac{2\pi\lambda}{1+\lambda\pi} \sin x, \quad \lambda \neq -\frac{1}{\pi}$$

if  $\lambda = -\frac{1}{\pi}$  there exist no solution

$$d) \quad f(x) = 1, \quad B_1 = \int_0^{2\pi} \cos x \, dx = 0$$

$$B_2 = \int_0^{2\pi} \sin x \, dx = 0$$

$$y = 1$$