MATH 544: METHODS OF APPLIED MATHEMATICS II
First Midterm Exam
March 14, 2016; 10.40-12.30

QUESTIONS: Choose any three of the following four problems

[35] **1**. **(a)** A functional is given as

$$J[y] = \int_0^{\ln 2} [y^2 + (y')^2] dx$$

with the boundary conditions $y(0) = 0$ and $y(\ln 2) = 3/4$. Find the function y which is the critical point of this functional. **(b)** Determine whether the critical point you found in part (a) minimizes or maximizes the functional $J[y]$.

Solution: **(a)**. Lagrange function is $L = y^2 + (y')^2$ and Euler-Lagrange equation gives

$$E(L) = \frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 2y - 2\,y'' = 0 \tag{1}$$

Solution of the resulting differential equation is

$$y(x) = A \sinh x + B \cosh x$$

where A and B are arbitrary constants. Using the boundary conditions we find

$$y(0) = 0 \quad \rightarrow \quad B = 0, \tag{2}$$
$$y(\ln 2) = 3/4 \quad \rightarrow \quad A = 1 \tag{3}$$

Then the solution reduces to $y(x) = \sinh x$.
(b). Any deviation from the extremum point is $y(x) = \sinh x + \varepsilon\, h(x)$ where $h(0) = h(\ln 2) = 0$ and h is an arbitrary twice differentiable function of $x \in [0, \ln 2]$. Then

$$J[y] = \int_0^{\ln 2} [y^2 + (y')^2] dx, \tag{4}$$

$$= \int_0^{\ln 2} [(\sinh x + \varepsilon\, h(x))^2 + (\cosh x + \varepsilon\, (h')^2\, dx, \tag{5}$$

$$= \int_0^{\ln 2} [(\cosh 2x + 2\varepsilon\, (h(x) \sinh x + h' \cosh x) + h^2 + (h')^2]\, dx, \tag{6}$$

$$= \int_0^{\ln 2} [(\cosh 2x + 2\varepsilon\, (h(x) \cosh x)' + h^2 + (h')^2]\, dx, \tag{7}$$

$$= J[y_0] + \int_0^{\ln 2} [h^2 + (h')^2]\, dx, \tag{8}$$

where $y_0 = \sinh x$. Then

$$J[\sinh x + \varepsilon\, h(x)] - J[\sinh x] \geq 0 \tag{9}$$

Hence $y = y_0 = \sinh x$ gives the minimum of the functional J.

[35] **2**. An initial value problem is given as follows: Is there a unique solution? If the answer is negative find conditions on data to have solutions.

$$\begin{aligned} &u_{tx} + u_t = 1, \quad t > 0, x \in \mathbb{R} \\ &u(x,0) = f(x), \quad x \in \mathbb{R} \\ &u_t(x,0) = g(x), \quad x \in \mathbb{R} \\ &u_x(x,0) = h(x), \quad x \in \mathbb{R} \end{aligned}$$

where f, g and h are given functions.

Solution: **(a)**. It is easy to solve the PDE. The general solution is

$$u(x,t) = t + F(x) + e^{-x}\, G(t) \tag{10}$$

where F and G are some arbitrary functions. The initial curve is $t = 0$ and since all $t = \textit{constant}$ curves are the characteristic curves of this differential equation, we expect that either the solutions do not exists or there are infinitely many solutions. Since

$$u_x(x,t) = F'(x) - e^{-x}\, G(t), \tag{11}$$

$$u_t(x,t) = 1 + e^{-x}\, G't \tag{12}$$

we get

$$\begin{aligned} u(x,0) &= F(x) + e^{-x}\, G(0) = f(x), \quad x \in \mathbb{R} && (13)\\ u_x(x,0) &= F'(x) - e^{-x}\, G(0) = h(x), \quad x \in \mathbb{R} && (14)\\ u_t(x,0) &= 1 + e^{-x}\, G'0 = g(x), \quad x \in \mathbb{R} && (15) \end{aligned}$$

The above conditions are not all independent. On the curve

$$\frac{du}{dt} = u_x \frac{dx}{dt} + u_y \frac{dy}{dt} = h(x) \tag{16}$$

which gives $f' = h(x)$. Hence not all $(f(x), g(x), h(x))$ are independent. Only two of them can be considered as independent.
Eq.(**??**) is the derivative of the Eq.(13), hence we have

$$F(x) = f(x) - e^{-x}\, G(0) \tag{17}$$

and the last equation (15) gives

$$e^{-x} + G'(0) = e^{-x}\, g(x) \tag{18}$$

Taking the derivative both sides we obtain

$$g(x) - g'(x) = 1 \tag{19}$$

This is constraint equation satisfied by the Cauchy data. Hence we have the following result: The given initial value problem has infinitely many solution if the constraint (19) is satisfied. The function F is is given in (17) but G is left arbitrary. Then the solutions are given by

$$u(x,t) = t + f(x) + e^{-x}\,(G(t) - G(0)) \tag{20}$$

If the condition (19) is not satisfied then the given initial value problem has no solution.

[35] **3**. An initial and boundary value problem is given as follows:

$$\begin{aligned} & u_{tt} - c^2\, u_{xx} = 0, \quad t > 0, \quad 0 < x < L\\ & u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x \le L\\ & u_x(0,t) = u_x(L,t) = 0, \quad t \ge 0, \end{aligned}$$

where f and g are given functions. Under what conditions this problem is a well posed problem?

Solution: This problem is similar to the problem we solved in class. The formal solution is

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} \cos(\lambda_n x)\,[a_n \cos(c\lambda_n t) + b_n \sin(c\lambda_n t)] \tag{21}$$

where $\lambda_n = \frac{n\pi}{L}$, $n =, 1, 2. \cdots$, a_n and b_n are the Fourier coefficients of the data, i.e.,

$$f(x) = a_0 + \sum_{n=1}^{\infty} \cos(\lambda_n x)\, a_n, \tag{22}$$

$$g(x) = \sum_{n=1}^{\infty} \cos(\lambda_n x)\, c\lambda_n\, b_n \tag{23}$$

The coefficients a_0, a_n and b_n are given by

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \tag{24}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\,\cos(\lambda_n x)\,dx, \quad n = 1, 2, \cdots \tag{25}$$

$$b_n = \frac{2}{cL\lambda_n}\int_0^L g(x)\,\cos(\lambda_n x)\,dx, \quad n = 1, 2, \cdots \tag{26}$$

For the existence of the solution, the infinite series corresponding to the second partial derivatives of $u(x,t)$ must be uniformally convergent. This implies that the series

$$\sum_{n=1}^{\infty} \lambda_n^2 |a_n|, \quad \sum_{n=1}^{\infty} \lambda_n^2 |b_n| \tag{27}$$

must be convergent. Hence both $a_n \approx \frac{1}{\lambda_n^4}$ and $b_n \approx \frac{1}{\lambda_n^4}$ These conditions lead to some restrictions on the data. Integrating the integrals in (25) and (26) by parts we obtain.

Theorem: The initial and boundary value problem has unique solution if $f'(0) = f'(L) = 0$, $f'''(0) = f'''(L) = 0$, and $f^{(iv)}(x)$ is integrable in $(0, L)$ and $g'(0) = g'(L) = 0$, and $g'''(x)$ is integrable in $(0, L)$.

The conditions on the data stated for the existence and uniqueness of the solutions are sufficient for the stability of this initial and boundary value problem. Furthermore these conditions guaranty the uniform convergence of the both series in (22) and (23). Hence this problem is well posed.

[35] **4**. If l is not preassigned, show that the stationary functions corresponding to the problem $\delta \int_0^l [y'^2 + 4(y - l)]dx = 0$, with $y(0) = 2$ and $y(l) = l^2$ are of the form $y = x^2 - 2(x/l) + 2$, where l is one of two real roots of the equation $2l^4 - 2l^3 - 1 = 0$. (Hint: This is a moving end point problem)

Solution: Functional is given by

$$J(y) = \int_0^\ell [(y')^2 + 4(y - \ell)]\, dx = \int_0^\ell [(y')^2 + 4y]\, dx - 4\ell^2 \tag{28}$$

With $y(0) = 2$ and $y(\ell) = \ell^2$. We can consider the end poind $(\ell, y(\ell))$ move on a curve $y = x^2$ (Note that in lecture 1 this means $g(x) = x^2$). Hence letting $L = (y')^2 + 4y$ we obtain

$$\delta J = \int_0^\ell E(L)h(x)dx + [L|_{x=\ell} - 8\ell]\delta\ell + \frac{\partial L}{\partial y'}|_{x=\ell}\delta h = 0 \tag{29}$$

Here $E(L) = 0$ which gives $y(x) = x^2 + ax + b$ from the boundary conditions we obtain

$$y(x) = x^2 - \frac{2}{\ell}x + 2 \tag{30}$$

Then from (29) we get

$$[4(\ell - \frac{1}{\ell})^2 + 4\ell^2 - 8\ell]\delta\ell + 4(\ell - \frac{1}{\ell})h(\ell) = 0 \tag{31}$$

The end point (ℓ, ℓ^2) is moving on the curve $y = g(x) = x^2$. A displacement of this point on this curve gives

$$y(\ell + \varepsilon\delta\ell) + \varepsilon h(\ell + \epsilon\delta\ell) = g(\ell + \varepsilon\delta\ell) = (\ell + \varepsilon\delta\ell)^2 \tag{32}$$

Differentiating this equation with respect to ε and then taking ε to zero we get (Lecture 1 page 14)

$$h(\ell) = [g'(\ell) - y'(\ell)]\delta\ell = \frac{2}{\ell}\,\delta\ell \tag{33}$$

When we use this identity in (31) we obtain the the constraint satisfied by ℓ

$$2\ell^4 - 2\ell^3 - 1 = 0 \tag{34}$$