MATH 544: METHODS OF APPLIED MATHEMATICS II

First Midterm Exam

March 24, 2014; 13.40-15.30

QUESTIONS:

[50] **1**. Find the Euler-Lagrange equation and also the natural boundary conditions for the functional $J(y) = \int_a^b L(x, y, y', y'')\, dx$ with the boundary conditions

$$y(a) = A_1, \quad y'(a) = A_2, \tag{1}$$

$$y(b) = B_1, \quad y'(b) = B_2 \tag{2}$$

where we assume $y \in C^4(a, b)$ and Lagrange function L is a twice differentiable function in $(a, b) \times \mathbb{R}^3$. Here $y' = \frac{dy}{dx}$ and A_1, A_2, B_1, B_2 are given real numbers. As an example find all y with $L = y\, y'^2 y''$. Discuss the case when that Lagrange function depends on y'' linearly, i.e., $L = y''\, f(y, y')$.

[50] **2**. An initial and boundary value problem is given as follows

$$\begin{aligned} &u_{tt} - u_{xx} = F(x,t), \quad t > 0,\ o < x < L \\ &u(0,t) = u(L,t) = 0,\ t \geq 0, \\ &u(x,0) = f(x), \quad u_t(x,0) = g(x),\ 0 \leq x \leq L \end{aligned}$$

where $f(x)$, $g(x)$ and $F(x, t)$ are given functions. Find condition(s) on the data $\{f(x), g(x), F(x, t)\}$ so that the problem is well posed.

Hint: The most general solution of the equation $y'' + k^2 y = h$ is

$$y(t) = a \cos kt + b \sin kt + \frac{1}{k^2} \int_0^t h(\tau)\, \sin k^2(t - \tau) d\tau, \quad k \neq 0$$

where a and b are arbitrary constants.