MATH 544: METHODS OF APPLIED MATHEMATICS II

Second Midterm Exam

April 30, 2013; 8.40-10.30

QUESTIONS:

[30]**1**. Consider

$$\nabla^2 u(r,\theta) = 0, \quad 0 \le r < a, \tag{1}$$

$$\frac{\partial u(a,\theta)}{\partial r} + \alpha\, u(a,\theta) = f(\theta), \tag{2}$$

with f is 2π periodic continuous function and $\alpha > 0$ a constant.

a. Solve formally by separation of variables.

b. Find reasonable conditions on f so that the formal solution is a solution.

[30]**2**. Suppose that $k > 0$ is a constant and $\nabla^2 u - k^2 u = 0$. Formulate Dirichlet, Neumann , and Robin problems of this operator $L = \nabla^2 - k^2$ and prove that each problem has at most one solution.

[60]**3**. Prove the following theorems.

Theorem 1 . *Let $u \in C^2(D) \cap C^1(\bar{D})$, where D is abounded domain.*

(i) If $\nabla^2 u \le 0$ in D and $u \ge 0$ on B. then $u \ge 0$ in D,

ii) If $\nabla^2 u \ge 0$ in D and $u \le 0$ on , B then $u \le 0$ in D.

Theorem 2. *Let $u \in C^2(D) \cap C^1(\bar{D})$, where D is a bounded domain.*

(i) If $\nabla^2 u \ge 0$ in D, then $u(x) \le M = max_{y\in B}\, u(y)$ for all $x \in D$.

(ii) If $\nabla^2 u \le 0$ in D, then $u(x) \ge m = \min_{y\in B}\, u(y)$ for all $x \in D$.

(iii) If$\nabla^2 u = 0$ in D, then $m \le u(x) \le M$ for all $x \in D$.

Theorem 3. *Let D be a bounded domain and B its boundary. Let $u \in C^2(D) \cap C^1(\bar{D})$. Then there is at most one solution of the problem*

$$\nabla^2 u = h, \quad in \quad D, \tag{3}$$

$$u = f, \quad on \quad B. \tag{4}$$

Theorem 4. *Suppose that $u \in C^2(D) \cap C^1(\bar{D})$ and satisfies the Dirichlet problem as in the previous theorem (Thm. 3). Suppose that $h(x)$ is bounded by a constant H in D and f is bounded by F on B. Then there is a constant C depending only on the domain D such that $|u(x)| \leq F + CH$*

Theorem 5. *Suppose that the functions $u_i \in C^2(D) \cap C^1(\bar{D})$, $i = 1, 2$, satisfies $\nabla^2 u_i = h_i$ in D and $u_i = f_i$ on B, where h_i and f_i are given. If*

$$|h_1(x) - h_2(x)| \leq \varepsilon$$

for all $x \in D$ and

$$|f_1(x) - f_2(x)| \leq \varepsilon$$

for all $x \in B$. then

$$|u_1(x) - u_2(x)| \leq (1 + C)\varepsilon$$

for all $x \in D$, where C is a constant determined by the domain D.