

MATH 544: METHODS OF APPLIED MATHEMATICS II

First Midterm Exam

March 19, 2013; 8.40-10.30

QUESTIONS:

[30] 1. An initial value problem is given as

$$\begin{aligned}u_{tt} - u_{xx} &= 0, \\u(x, 0) &= f(x), \quad u_t(x, 0) = g(x)\end{aligned}$$

where $f(x)$ and $g(x)$ are given function. Find condition(s) on $f(x)$ and $g(x)$ so that the problem has solutions.

[30] 2. An initial and boundary value problem is given as

$$\begin{aligned}u_{tt} - c^2 u_{xx} &= 0, \quad -L < x < L, \quad t > 0, \\u_x(0, t) &= u_x(L, t) = 0, \quad t \geq 0, \\u(x, 0) &= f(x), \quad u_t(x, 0) = 0, \quad -L \leq x \leq L\end{aligned}$$

where $f(x)$ is a given function. Find conditions on $f(x)$ so that the problem is well defined.

[30] 3. Find the curve(s) for which the functional

$$J[y] = \int_0^a \frac{\sqrt{1+y'^2}}{y} dx, \quad y(0) = 0, \quad (1)$$

have extrema if the point (a, b) vary along the line $y = x - 5$

[30] 4. Find the extrema of the problem

$$\delta \int_0^\pi (y'')^2 dx = 0, \quad y(0) = y''(0) = 0, \quad y(\pi) = y'(\pi) = 0 \quad (2)$$

with

$$\int_0^\pi y^2 dx = 1$$

(1)

$$(1) \quad u_{tt} - u_{xx} = 0$$

General solution: $u(x,t) = F(x+t) + G(x-t)$
 where F and G are arbitrary solutions

Initial conditions:

$$u(x,0) = F(2x) + G(0) = f(x)$$

$$u_t(x,0) = F_\xi(2x) - G_\eta(0) = g(x) \quad \xi = x+t, \eta = x-t$$

$$\Rightarrow F(2x) + G(0) = f(x)$$

$$\frac{1}{2} F_x(2x) - G_\eta(0) = g(x)$$

$$\frac{1}{2} [f_x - G_\eta(0)]_x - G_\eta(0) = g(x)$$

$$\Rightarrow f_x(x) - 2G_\eta(0) = 2g(x)$$

$$\text{or } f_{xx} = 2g_x$$

- If this condition is not satisfied the above IVP problem does not have solution.
- If this condition is satisfied by the data

$$\Rightarrow F(x) = f(x/2) - G(0)$$

and $G(x)$ is left arbitrary

and

$$u(x,t) = f\left(\frac{x+t}{2}\right) - G(0) + G(x-t)$$

Infinitely many solutions.

(2) using the separation of variable.

$$u(x,t) = X(x)T(t).$$

we get

$$c^2 \frac{X''}{X} = \frac{T''}{T} = -\mu c^2$$

$X(x)$ satisfies the following BVP problem.

$$\left. \begin{aligned} X'' + \mu X &= 0 \\ X'(0) &= 0 \\ X'(L) &= 0 \end{aligned} \right\}$$

The only possibility (nontrivial solution) $\mu = \lambda_n^2 > 0$

$$X(x) = A_n \cos(\lambda_n x), \quad \lambda_n = \frac{n\pi}{L}$$

$$\Rightarrow u(x,t) = \sum [A_n \sin(\lambda_n c t) + B_n \cos(\lambda_n c t)] \cos(\lambda_n x)$$

Initial cond. $u_t(x,0) = 0 \Rightarrow A_n = 0$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos(\lambda_n c t) \cos(\lambda_n x) \quad (1)$$

$$\text{where } u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \cos \lambda_n x \quad (2)$$

$$\text{with } B_n = \frac{2}{L} \int_0^L f(x) \cos \lambda_n x \, dx \quad n=1,2, \dots \quad (3)$$

• Existence and uniqueness: (1) must satisfy the wave eqn. Hence (2)

$$\sum_{n=1}^{\infty} \lambda_n^2 (|B_n|) K \text{ must converge}$$

$$\text{or } |B_n| \sim \frac{1}{\lambda_n^2}$$

$\Rightarrow$ using integration by part in (3) we get

$$B_n = \frac{2}{L \lambda_n^4} \int_0^L f^{(4)} \cos(\lambda_n x) dx$$

$$\text{with } f'(0) = f'(L) = 0$$

$$f'''(0) = f'''(L) = 0$$

$$\text{and } f^{(4)}(x) \text{ cont. on } [0, L]$$

$$\begin{aligned} \Rightarrow \sum \lambda_n^2 |B_n| &= \sum \lambda_n^2 \frac{2}{L \lambda_n^4} M = \frac{2}{L} M \sum \frac{1}{\lambda_n^2} \\ &= \frac{2M}{L} \frac{L^2}{\pi^2} \sum \frac{1}{n^2} = \frac{2ML}{\pi^2} \pi^2/6 \\ &= \frac{ML}{3} \end{aligned}$$

Hence (1) is uniformly conv. on $x \in [0, L]$ and $t \geq 0$.
The above conditions on $B_n \sim \frac{1}{\lambda_n^2}$ under (2) unif. conv. $\Rightarrow$ The initial and BV problem has a unique solution given by (1) and (3).

(9)

stability: Same as done in class

$$(3) \quad L = \sqrt{1+y'^2}/y, \quad y(0)=0 \\ g(x) = x-5$$

$$i) \quad E(L) = 0$$

$$ii) \quad \frac{\partial L}{\partial y'} + \frac{L}{g'-y'} = 0 \quad \text{at } x=a.$$

$$E(L) = -\frac{1}{y^2} \sqrt{1+y'^2} - \frac{d}{dx} \left(\frac{1}{y} \frac{y'}{\sqrt{1+y'^2}} \right) = 0 \\ = -\frac{1}{y^2} \sqrt{1+y'^2} + \frac{1}{y^2} \frac{y'^2}{\sqrt{1+y'^2}} - \frac{y''}{y \sqrt{1+y'^2}}$$

$$+ \frac{y'^2 y''}{y (1+y'^2)^{3/2}} = 0$$

$$- (1+y'^2)^2 + y'^2 (1+y'^2) - y (1+y'^2) y'' \\ + y y'^2 y'' = 0$$

$$-1 - 2y'^2 - y'^4 + y'^2 + y'^4 - y y'' = 0$$

$$-1 - y'^2 - y y'' = 0 \Rightarrow (y y')' + 1 = 0$$

$$y y' + x = A$$

$$\boxed{y^2 + x^2 = 2Ax + B} \quad A, B \text{ are constants}$$

$$ii) \quad \frac{\partial L}{\partial y'} = \frac{y'}{y\sqrt{1+y'^2}}$$

$$\frac{\partial L}{\partial y'} + \frac{L}{g'-y'} = \frac{y'}{y\sqrt{1+y'^2}} + \frac{\sqrt{1+y'^2}}{y(1-y')} = 0$$

$$y'(1-y') + 1+y'^2 = 0$$

$$y' = -1 \quad \text{at } x=a$$

$$\text{at } x=a \quad y(a) = g(a) = a-5.$$

$$\text{and} \quad y^2 + x^2 = 2Ax$$

$$yy' + x = A$$

$$\text{at } x=a \quad -y_a + a = A.$$

$$\text{since } y_a = a-5 \Rightarrow A=5$$

This completes the determination of the curve

$$y^2 + x^2 = 10x$$

$$\underline{y = \pm \sqrt{10x - x^2}}$$

(6)

$$41 \quad L = y''^2 + \lambda y^2$$

$$E(u) = 2\lambda y + \frac{d^2}{dx^2} (2y'') = 0$$

$$y'''' + \lambda y = 0$$

$$\text{let } \lambda = \mu^2 > 0$$

$$y(x) = A \cos \sqrt{\mu} x + B \sin \sqrt{\mu} x + C \cosh \sqrt{\mu} x + D \sinh \sqrt{\mu} x$$

A, B, C, D are constant.

$$BC: \quad y(0) = A + C$$

$$y'(0) = -\mu A + \mu C = 0$$

$$A = C = 0$$

$$y(x) = B \sin \sqrt{\mu} x + D \sinh \sqrt{\mu} x$$

$$y(\pi) = B \sin \sqrt{\mu} \pi + D \sinh \sqrt{\mu} \pi = 0$$

$$y'(\pi) = -B \sqrt{\mu} \cos \sqrt{\mu} \pi + D \mu \cosh \sqrt{\mu} \pi = 0$$

$$D = 0$$

$$\sqrt{\mu} \pi = n\pi \Rightarrow \mu = n^2 \quad \lambda = n^4$$

$$y(x) = B \sin(nx)$$

$$B^2 \int_0^\pi \sin^2(nx) dx = 1 \Rightarrow B^2 \int_0^\pi \left(\frac{1 - \cos 2nx}{2} \right) dx$$

$$\begin{aligned} \cos 2nx &= 2 \cos^2 nx - 1 \\ &= 1 - \sin^2 nx \end{aligned}$$

$$B^2 \frac{\pi}{2} = 1$$

$$y(x) = \sqrt{\frac{2}{\pi}} \sin(nx) \quad , \quad n = 1, 2, \dots$$

$$\lambda = n^4$$