

solutions of the VIth homework
problems

HOMEWORK VI (MATH 544): April 18, 2012
(For May 2, 2012)

1.8. In two dimensions consider

$$\nabla^2 u = 0, \quad 0 < x < a, \quad 0 < y < b, \quad (1)$$

$$u(x, 0) = f(x), \quad u(x, b) = 0, \quad 0 \leq x \leq a, \quad (2)$$

$$u(0, y) = u(a, y) = 0, \quad 0 \leq y \leq b. \quad (3)$$

- a. Find the formal solution,
- b. Find reasonable restrictions on $f(x)$ so that the formal solution is a solution.

2. Find Green's function of the following problem

$$-k u'' + l u = 0, \quad u(0) - u'(0) = 0, \quad u(1) = 0.$$

Find the solution of the boundary value problem.

3. Show that the Green's function is unique, if it exists. *Hint.* If both $G(x, y)$ and $H(x, y)$ provides the same $u(x)$ given by

$$u(x) = \int_a^b G(x, y) f(y) dy, \quad (4)$$

$$u(x) = \int_a^b H(x, y) f(y) dy, \quad (5)$$

then

$$\int_a^b [G(x, y) - H(x, y)] f(y) dy = 0$$

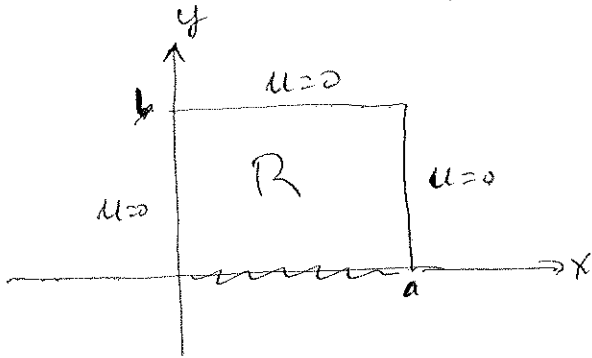
Homework VI

①

problem 1. $\nabla^2 u = 0$, $0 < x < a$, $0 < y < b$

$$u(x, 0) = f(x), \quad u(x, b) = 0 \quad 0 \leq x \leq a.$$

$$u(0, y) = u(a, y) = 0, \quad 0 \leq y \leq b.$$



Soln: $u(x, y) = \sum_{n=1}^{\infty} a_n \sinh \lambda_n (y-b) \cdot \sin(\lambda_n x)$
($\lambda_n = \frac{n\pi}{a}, n=1, 2, \dots$)

$$u(x, 0) = f(x) = - \sum_{n=1}^{\infty} a_n \sinh(\lambda_n b) \sin \lambda_n x$$

$$\Rightarrow a_n \sinh \lambda_n b = - \frac{2}{a} \int_0^a \sin(\lambda_n x) f(x) dx$$

Justification: series corresponding to u_{xx} or u_{yy} is

$$- \sum_{n=1}^{\infty} a_n \lambda_n^2 \sinh \lambda_n (y-b) \sin(\lambda_n x) \quad \text{which should be}$$

uniformly convergent

$$\left| \sum a_n \lambda_n^2 \sinh \lambda_n (y-b) \sin \lambda_n x \right| \leq \sum |a_n| \lambda_n^2 |\sinh \lambda_n (y-b)|$$

~~$$\leq \sum |a_n| \lambda_n^2 \sinh \lambda_n b$$~~

$$\leq \frac{2}{a} \sum \lambda_n^2 \left| \frac{\sinh \lambda_n (y-b)}{\sinh \lambda_n b} \right| \int_0^a |f(x)| dx$$

$$0 < y < b$$

$$\begin{aligned} \frac{\sinh \lambda_n (y-b)}{\sinh \lambda_n b} &= \frac{e^{\lambda_n (y-b)} - e^{-\lambda_n (y-b)}}{e^{\lambda_n b} - e^{-\lambda_n b}} \\ &= \frac{e^{\lambda_n y - 2\lambda_n b} - e^{-\lambda_n y}}{1 - e^{-2\lambda_n b}} \end{aligned}$$

$$1 - e^{-2\lambda_n b} \leq 1$$

$$e^{-2\lambda_n b} \leq e^{-2\lambda_1 b}$$

$$-e^{-2\lambda_1 b} \leq -e^{-2\lambda_n b}$$

$$1 - e^{-2\lambda_1 b} \leq 1 - e^{-2\lambda_n b}$$

$$\frac{1}{1 - e^{-2\lambda_n b}} \leq \frac{1}{1 - e^{-2\lambda_1 b}}$$

$$\Rightarrow \frac{\sinh \lambda_n (y-b)}{\sinh \lambda_n b} \leq \frac{e^{\lambda_n y - 2\lambda_n b} - e^{-\lambda_n y}}{1 - e^{-2\lambda_1 b}}$$

~~One can prove that~~
we showed that

$$\begin{aligned}
 \left| \frac{\sinh \lambda_n (y-b)}{\sinh \lambda_n b} \right| &\leq \frac{1}{1-e^{-2\lambda_n b}} \left| e^{\lambda_n y - 2\lambda_n b} - e^{-\lambda_n y} \right| \\
 &\leq \frac{1}{1-e^{-2\lambda_n b}} \left| e^{\lambda_n y - 2\lambda_n b} - e^{-\lambda_n b} \right| \\
 &\leq \frac{1}{1-e^{-2\lambda_n b}} e^{-\lambda_n b} \left| e^{\lambda_n (y-b)} - 1 \right| \\
 &\leq \frac{2}{1-e^{-2\lambda_n b}} e^{-\lambda_n b} \quad \forall y \in (0, b).
 \end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
 \left| \sum a_n \lambda_n^2 \sinh \lambda_n (y-b) \sin \lambda_n x \right| &\leq \frac{4 \int_0^a |f(x)| dx}{a(1-e^{-2\lambda_1 b})} \sum_{n=1}^{\infty} \lambda_n^2 e^{-\lambda_n b} \\
 &\leq \frac{4}{a(1-e^{-2\lambda_1 b})} \left(\int_0^a |f(x)| dx \right) \underbrace{\sum_{n=1}^{\infty} \lambda_n^2 e^{-\lambda_n b}}_{\text{convergent}}.
 \end{aligned}$$

Hence the series corresponding to the partial derivatives u_{xx}, u_{yy} are uniformly convergent if $|f(x)|$ is integrable in $(0, a)$.

For the initial condition to be satisfied

(4)

$$u(x,0) = f(x) = -\sum a_n \sinh(\lambda_n b) \sin(\lambda_n x)$$

with

$$a_n \sinh \lambda_n b = -\frac{2}{a} \int_0^a \sin(\lambda_n x) f(x) dx$$

$$f(0) = f(a) = 0$$

f'' is continuous in $(0,a)$ then

the above series is uniformly convergent.

$$2) -k u'' + \ell u = 0 \quad x \in (0,1)$$

$$u(0) - u'(0) = 0$$

$$u(1) = 0$$

Solution: $u'' = \frac{\ell}{k} u, \quad k \neq 0$

$$i) \quad \frac{\ell}{k} = \lambda^2 \quad u = a e^{\lambda x} + b e^{-\lambda x}$$

$$u(1) = a e^{\lambda} + b e^{-\lambda} = 0 \Rightarrow b = -a e^{2\lambda}$$

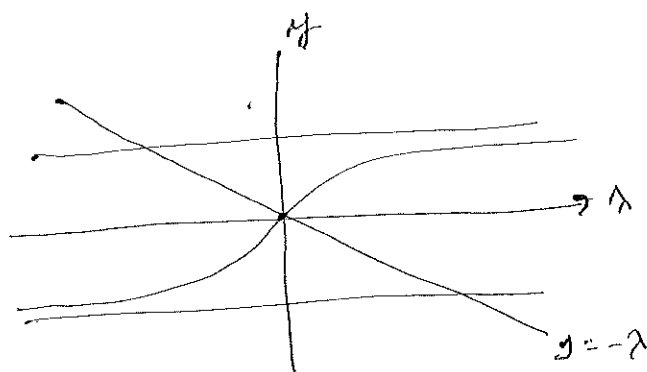
$$\Rightarrow u(x) = a (e^{\lambda x} - e^{2\lambda - \lambda x}) = a (e^{\lambda x} - e^{\lambda(2-x)})$$

$$= a e^{\lambda} (e^{\lambda(x-1)} - e^{\lambda(1-x)})$$

$$= 2 a e^{\lambda} \sinh \lambda(x-1)$$

$$u(0) - u'(0) = -\sinh \lambda - 2 \cosh \lambda = 0$$

$$\tanh \lambda = -\lambda$$



$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

one intersection pt at $\lambda = 0$
where we have a trivial
solution.

Hence $\frac{\ell}{k} = \lambda^2$ is not possible

$$\text{ii) } \frac{L}{u} = -\lambda^2$$

$$u = a \cos \lambda x + b \sin \lambda x.$$

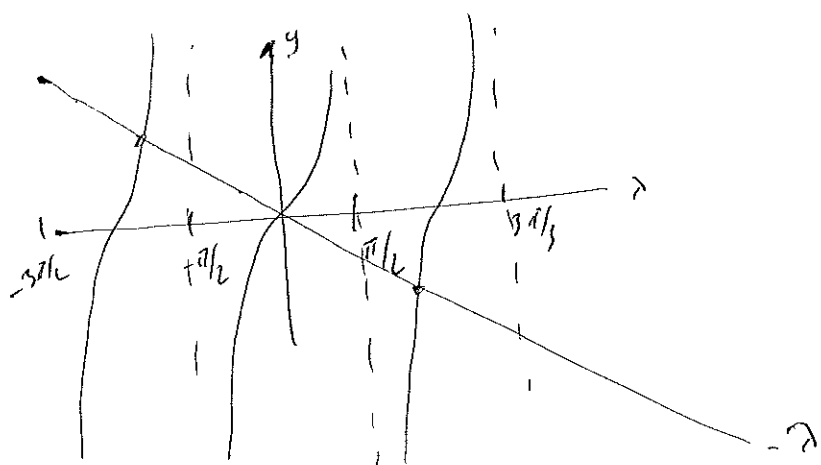
$$u(1) = a \cos \lambda + b \sin \lambda = 0$$

$$\lambda \neq 0$$

$$b = -\frac{a \cos \lambda}{\sin \lambda}$$

$$u(x) = \frac{a}{\sin \lambda} [\cos \lambda x \sin \lambda - \sin \lambda x \cos \lambda] = -\frac{a}{\sin \lambda} \sin \lambda (x-1)$$

$$u(0) - u'(0) = -\frac{a}{\sin \lambda} [-\sin \lambda - \lambda \cos \lambda] = 0 \Rightarrow \tan \lambda = -\lambda$$



Infinitely many intersections
hence infinitely many solutions λ_n

$$u = A_n \sin \lambda_n (x-1)$$

A_n 's are constant.

This BVP is an eigenvalue eqn. with eigenvalues λ_n and eigenfunctions $\sin \lambda_n (x-1)$ ($0 < x < 1$).

ii) Green's function approach: Take L operator as $\frac{d^2}{dx^2}$

(7)

$$(L + \lambda^2)u = 0$$

$$L G(x, y) = \delta(x, y)$$

$$\int_0^1 (G L u - u L G) dx = (G u' - u G') \Big|_0^1 = 0$$

Because G satisfies the same BCs as u .

$$-\lambda^2 \int_0^1 u(x) G(x, y) dx = u(y) = 0$$

$$\text{or } u(x) = -\lambda^2 \int_0^1 u(y) G(x, y) dy$$

Fredholm integral equation of the second kind, and it is homogeneous. Since $L = L^T \Rightarrow G(x, y) = G(y, x)$ a symmetric kernel. One can find $G(x, y)$ explicitly

$$G(x, y) = \begin{cases} \frac{1}{2}(1+x)(1-y) & x \leq y \\ -\frac{1}{2}(1-x)(1+y) & x > y \end{cases}$$

$$G(1) = 0$$

$$G'(0) = G'(1)$$

$$G(x=y) = G_y(x=y)$$

$$G'_+ - G'_- = 1$$

$$u(x) = +\lambda^2 \left[\frac{1}{2} \int_0^x (1-x)(1+y) u(y) dy + \frac{1}{2} \int_x^1 (1+x)(1-y) u(y) dy \right]$$

$$= -\frac{1}{2} \lambda^2 \int_0^x (x-1)(1+y) u(y) dy + \frac{1}{2} \lambda^2 \int_x^1 (1+x)(y-1) u(y) dy$$

Now show that

$$u(x) = C \sin \lambda (x-1)$$

where $\tan \lambda = -\lambda$

proof

$$\int_0^x (1+y) \sin \lambda (y+1) dy = -\frac{1}{\lambda} \cos \lambda (y+1) (1+y) \Big|_0^x + \frac{1}{\lambda} \int_0^x \cos \lambda (y+1) dy$$

$$= -\frac{1+x}{\lambda} \cos \lambda (x+1) + \frac{1}{\lambda} \cos \lambda + \frac{1}{\lambda^2} \sin \lambda (y+1) \Big|_0^x$$

$$= -\frac{1+x}{\lambda} \cos \lambda (x+1) + \frac{\cos \lambda}{\lambda} + \frac{1}{\lambda^2} \sin \lambda (x+1) + \frac{\sin \lambda}{\lambda^2}$$

$$\int_x^1 (y-1) \sin \lambda (y-1) dy = -\frac{1}{\lambda} (y-1) \cos \lambda (y-1) \Big|_x^1 + \frac{1}{\lambda} \int_x^1 \cos \lambda (y-1) dy$$

$$= +\frac{1}{\lambda} (x-1) \cos \lambda (x-1) - \frac{1}{\lambda^2} \sin \lambda (x-1)$$

$$u(x) = -\frac{C}{2} \lambda^2 (x-1) \left[-\frac{1+x}{\lambda} \cos \lambda (x+1) + \frac{1}{\lambda} \cos \lambda + \frac{1}{\lambda^2} \sin \lambda (x+1) + \frac{\sin \lambda}{\lambda^2} \right]$$

$$- \frac{C}{2} \lambda^2 (1+x) \left[\frac{1}{\lambda} (x-1) \cos \lambda (x-1) - \frac{1}{\lambda^2} \sin \lambda (x-1) \right]$$

$$= -\frac{C}{2} (x-1-x-1) \sin \lambda (x-1) = C \sin \lambda (x-1) \quad \checkmark$$

Hence

The Fredholm eqn. of the second kind.

$$u(x) = -\lambda^2 \int_0^1 u(y) G(x, y) dy$$

has infinitely many eigenvalues λ_n and eigenfunctions $\sin \lambda_n(x)$ for the symmetric kernel

$$G(x, y) = - \begin{cases} \frac{1}{2} (1+x)(1-y) & x \leq y \\ \frac{1}{2} (1-x)(1+y) & x > y \end{cases}$$

and λ_n 's are the solutions of $\tan \lambda_n = -\lambda_n$
 $n = 1, 2, \dots$