solutions

MATH 543

METHODS OF APPLIED MATHEMATICS I

Final Exam

January 2, 2020, Friday 15.30-17.30, SA-Z03

PROBLEMS: *There are five questions in this exam. Solve as many problems as you wish. The total number of points must be at least 100.*

[30]**1.** Solve the following initial value problem

$$y'' + 3y' + 2y = f(x), \quad x > 0,$$
$$y(0) = 0, \quad y'(0) = 0$$

where f is a continuous function over $[0, \infty)$

[30]**2.** Let $z = z_0$ be a regular singular point of the differential equation $y'' + p(z)\,y' + q(z)\,y = 0$. If the difference $r_1 - r_2$ of the indices r_1 and r_2 of the differential equation is a positive integer or zero, prove that the solution of the differential equation about z_0 contains logarithmic singularity. (see the LN)

[30]**3.** Find a uniformly valid first order approximate solution of

$$y'' + y' + \varepsilon\, y^2 = 0, \quad t > 0, \quad \varepsilon << 1,$$
$$y(\infty) = 0, \quad y'(0) = 0,$$

[30]**4.** Find the dominant terms of the following integrals. In each case $\lambda >> 1$

$$(a) \quad I_1(\lambda) = \int_0^{\pi} e^{\lambda \sin^2 t}\, dt,$$
$$(b) \quad I_2(\lambda) = \int_{\lambda}^{\infty} e^{-t^2}\, dt$$

[30]**5.** Find a uniformly valid approximate solution of

$$\varepsilon\, y'' + 2y' + y = 0, \quad 0 < t < 1, \quad \varepsilon << 1,$$
$$y(0) = 0, \quad y(1) = 1$$

problem 1 Solve the initial value problem

$$y'' + 3y' + 2y = f(x), \quad x > 0$$

$$y(0) = 0, \quad y'(0) = 0$$

where $f(x)$ is a continuous function over $[0, \infty)$

Solution: Fundamental solution of the homogeneous equation are $y_1(x) = e^{-x}$, $y_2(x) = e^{-2x}$. Hence the Greens function of the problem is

$$G(x,y) = \begin{cases} a_1 e^{-x} + b_1 e^{-2x}, & x \leq y \\ a_2 e^{-x} + b_2 e^{-2x}, & x \geq y \end{cases}$$

Greens function satisfies the initial condition

$$\left. \begin{array}{l} a_1 \cancel{+} b_1 = 0 \\ -a_1 - 2b_1 = 0 \end{array} \right\} a_1 = b_1 = 0$$

Hence

$$G(x,y) = \begin{cases} 0, & x \leq y \\ a_2 e^{-x} + b_2 e^{-2x}, & x \geq y \end{cases}$$

continuity at $x = y$

$$a_2 e^{-y} + b_2 e^{-2y} = 0 \Rightarrow a_2 = -b_2 e^{-y}$$

Jump condition at $x = y$

$$-a_2 e^{-y} - 2b_2 e^{-2y} = 1$$

Then we find $b_2 = -e^{2y}$, $a_2 = e^{y}$

$\Rightarrow$

$$G(x,y) = \begin{cases} 0 , & x \le y \\ e^{y-x} - e^{2y-2x} , & x \geq y \end{cases}$$

Then the soluh

$$y(x) = \int_0^x (e^{y-x} - e^{2y-2x}) f(y)\, dy$$

$$= e^{-x} \int_0^x e^{y} f(y)\, dy - e^{-2x} \int_0^x e^{2y} f(y)\, dy$$

$y(0) = 0$,

$$y'(x) = \int_0^x (-e^{y-x} + 2e^{2y-2x}) f(y)\, dy$$

$y'(0) = 0$

$$y''(x) = f(x) + \int_0^x [e^{y-x} - 4e^{2y-2x}] f(y)\, dy$$

$$y'' + 3y' + 2y = f + \int_0^x (\cancel{e^{y-x}} - \cancel{4e^{2y-2x}}) f\, dy$$

$$+ \int_0^x (-\cancel{3e^{y-x}} + \cancel{6e^{2y-2x}}) f\, dy$$

$$+ \int_0^x (\cancel{2e^{y-x}} - \cancel{2e^{2y-2x}}) f\, dy = f \checkmark$$

Problem 2: See the Lecture Notes

Problem 3 Find a first order approximate solution of the initial value problem

$$y'' + y' + \varepsilon y^2 = 0, \quad t > 0, \quad \varepsilon \ll 1$$

$$y'(0) = 1, \quad y(\infty) = 0$$

Solution: let

$$y(t, \varepsilon) = y_0(t) + \varepsilon y_1(t) + \cdots$$

i) $y_0'' + y_0' = 0, \quad y_0'(0) = 1, \quad y_0(\infty) = 0$

$$y_0 = -e^{-t}$$

ii) $y_1'' + y_1' + y_0^2 = 0 \qquad y_1'(0) = 0$

$\qquad y_1(\infty) = 0$

$$y_1'' + y_1' + e^{-2t} = 0$$

$$(y_1' e^t) + e^{-t} = 0$$

$$y_1' e^t - e^{-t} = c_1$$

$$y_1' = c_1 e^{-t} + e^{-2t}$$

$$y_1(t) = -c_1 e^{-t} - \frac{1}{2} e^{-2t} + c_2$$

$y_1(\infty) = 0 \Rightarrow c_2 = 0$

$y_1'(0) = c_1 + 1 = 0 \qquad c_1 = -1$

$$y_1(t) = e^{-t} - \frac{1}{2} e^{-2t}$$

$$y_{cpp} = y_0 + \varepsilon y_1$$

$$= -e^{-t} + \varepsilon \left(e^{-t} - \tfrac{1}{2} e^{-2t} \right)$$

$$y_{cp}(\infty) = 0 \quad ✓$$

$$y'_{cp}(0) = 1 \quad ✓$$

$$y'_{cpp} = e^{-t} + \varepsilon \left(-e^{-t} + e^{-2t} \right)$$

$$y''_{cpp} = -e^{-t} + \varepsilon \left(e^{-t} - 2e^{-2t} \right)$$

$$r(t,\varepsilon) = y''_{cp} + y'_{cp} + \varepsilon y_{cp}^2$$

$$= \cancel{e^{-t}} + \varepsilon \left(\cancel{-e^{-t}} + \cancel{e^{-2t}} \right) - \cancel{e^{-t}} + \varepsilon \left(\cancel{e^{-t}} - \cancel{2e^{-2t}} \right)$$

$$+ \varepsilon \left(\cancel{e^{-2t}} + 2\varepsilon \left(e^{-2t} - \tfrac{1}{2} e^{-3t} \right) + \varepsilon^2 \left(e^{-t} - \tfrac{1}{2} e^{-4} \right)^2 \right)$$

$$= 2\varepsilon^2 \left(e^{-2t} - \tfrac{1}{2} e^{-3t} \right)$$

$$+ \varepsilon^3 \left(e^{-t} - \tfrac{1}{2} e^{-2t} \right)^2$$

$$|r(t,\varepsilon)| \leq 2\varepsilon^2 \left(1 + 1/2 \right) + \varepsilon^3 \left(1 + \tfrac{1}{2} \right)^2$$

$$= 3\varepsilon^2 + \tfrac{9}{4} \varepsilon^3$$

Hence the residue function $r(t,\varepsilon)$

$$|r(t,\varepsilon)| \le 3\varepsilon^2 + \frac{9}{4}\varepsilon^3$$

for all $t \geq 0$

then $\lim_{\varepsilon \to 0} r(t,\varepsilon) = 0$ uniformly

over $[0,\infty)$. Then our approximate solution is a uniformly valid approximation

Problem 4

a) $I_1 = \int_0^{\pi} e^{\lambda \sin^2 t}\, dt \quad , \quad \lambda \gg 1$

$\sin t$ has its maximum value at $t = \pi/2$

$\Rightarrow \quad \sin^2 t = 1 - (t - \pi/2)^2 + O((t - \pi/2)^3)$

$$I_1(\lambda) \simeq \int_0^{\pi} e^{\lambda - \lambda (t - \pi/2)^2}\, dt$$

$$\simeq e^{\lambda} \int_0^{\pi} e^{-\lambda (t - \pi/2)^2}\, dt$$

let $\sqrt{\lambda}\,(t - \pi/2) = u \quad \Rightarrow$

$$I_1(\lambda) \simeq \frac{e^{\lambda}}{\sqrt{\lambda}} \underbrace{\int_{-\sqrt{\lambda}\,\pi/2}^{\sqrt{\lambda}\,\pi/2} e^{-u^2}\, du}_{\simeq \sqrt{\pi}}$$

$$\simeq \sqrt{\frac{\pi}{\lambda}}\, e^{\lambda}$$

b) $I_2(\lambda) = \int_{\lambda}^{\infty} e^{-t^2}\, dt$

let $t = \lambda + s \quad \Rightarrow$

$$I_2(\lambda) = \int_0^{\infty} e^{-\lambda^2 - 2\lambda s - s^2}\, ds$$

$$I_2(\lambda) = e^{-\lambda^2} \int_0^\infty e^{-2\lambda t} e^{-t^2} dt$$

$$2t = \tau \qquad dt = \frac{1}{2} d\tau$$

$$I_2(\lambda) = e^{-\lambda^2} \int_0^\infty e^{-\lambda \tau} e^{-\tau^2/4} dt$$

$$= e^{-\lambda^2} \int_0^\infty e^{-\lambda t} \left(1 - t^2/4 + \dots\right)$$

$$= e^{-\lambda^2} \left[\int_0^\infty e^{-\lambda t} dt - \frac{1}{4} \int_0^\infty e^{-\lambda t} t^2 dt \right]$$

$$= e^{-\lambda^2} \left[\frac{1}{\lambda} + O\left(\frac{1}{\lambda^3}\right) \right]$$

$$\simeq \frac{1}{\lambda} e^{-\lambda^2}$$

problem 5. Find a uniformly valid approximate solution of

$$\varepsilon y'' + 2y' + y = 0 \quad , 0 < t < 1 \quad , \quad \varepsilon << 1$$

$$y(0) = 0 \, , \; y(1) = 1$$

Solution: We use singular perturbation method

a) Outer solution (we stay at the zeroth order perturbation)

$$y_{out}(t, \varepsilon) = y_0(t) + \varepsilon y_1(t) + \cdots$$

$$2y_0' + y_0 = 0 \quad , y_0(0) = 0 \, , \; y_0$$

$$y_0(t) = A e^{-t/2} \quad , \; y_0(1) = A e^{-1/2} = 1$$

$$A = e^{1/2}$$

$$y_{out}(t) = e^{\frac{1}{2}(1-t)}$$

b) Inner soln: let $t = \delta \tau$

$$\Rightarrow \quad \frac{\varepsilon}{\delta^2} y_{\tau\tau} + \frac{2}{\delta} y_\tau + y = 0$$

only possible case is $\delta = \varepsilon$

$$\Rightarrow \quad y_{\tau\tau} + 2 y_\tau + \varepsilon y = 0$$

$$y(\tau, \varepsilon) = y_0(\tau) + \varepsilon y_1(\tau) + \cdots$$

$$y_{0\tau\tau} + 2 y_{0\tau} = 0$$

$$y_0(\tau) = A_1 + B_1 e^{-2\tau}$$

$$y_0(0) = 0 \Rightarrow B_1 = -A_1$$

$$y_0(\tau) = A_1\left(1 - e^{-2t/\varepsilon}\right)$$

c) Matching zone condition: let $t = \sqrt{\varepsilon}\, y$

$$\Rightarrow \lim_{\varepsilon \to 0} y_{out} = e^{1/2}$$

$$\Rightarrow A_1 = e^{1/2}$$

$$\Rightarrow \lim_{\varepsilon \to 0} y_{in} = A_1$$

$$y_{in} = e^{1/2} - e^{\frac{1}{2} - 2t/\varepsilon}$$

d) Approximate solution

$$y_{app}(t,\varepsilon) = y_{out} + y_{in} - e^{1/2}$$

$$= e^{\frac{1}{2}(1-t)} - e^{\frac{1}{2} - 2t/\varepsilon}$$

e) The residue function

$$r(t,\varepsilon) = \varepsilon y''_{app} + 2y'_{app} + y_{ap}$$

(10)

$$y'_{app} = -\frac{1}{2} e^{\frac{1}{2}(1-t)} + 2/_\varepsilon\, e^{\frac{1}{2}-\frac{2t}{\varepsilon}}$$

$$y''_{app} = \frac{1}{4} e^{\frac{1}{2}(1-t)} - \frac{4}{\varepsilon^2} e^{\frac{1}{2}-2t/\varepsilon}$$

$$\varepsilon y''_{app} + 2 y'_{ap} + y_{ap}$$

$$= \frac{\varepsilon}{4} e^{\frac{1}{2}(1-t)} - \cancel{\frac{\varepsilon 4}{\varepsilon} e^{\frac{1}{2}-2t/\varepsilon}}$$

$$- \cancel{e^{1/2(1-t)}} + \cancel{\frac{4}{\varepsilon} e^{1/2-2t/\varepsilon}}$$

$$+ \cancel{e^{\frac{1}{2}(1-t)}} - e^{\frac{1}{2}-2t/\varepsilon}$$

$$r(t,\varepsilon) = \frac{\varepsilon}{4} e^{\frac{1}{2}(1-t)} - e^{\frac{1}{2}-2t/\varepsilon}$$

let $t \in [0,T]$ where T is a positive real number then

$\lim_{\varepsilon\to 0} r(t,\varepsilon) = 0$ uniformly over $[0,T]$

and $y(0) = 0$ ✓, $y(1) = 1 - e^{1/2-\frac{2}{\varepsilon}}$

$= 1 - O(\varepsilon^n)$ ✓

Then $y_{app.}$ is a uniformly valid approximation