SOLUTIONS

MATH 543

METHODS OF APPLIED MATHEMATICS I
Second Midterm Exam

December 13, 2019, Friday 13.40-15.30, SA-Z19

PROBLEMS: *There are four questions in this exam. Solve as many problems as you wish. The total number of points must be at least 100.*

[35]1. Consider $u''+u=f(x)$ with $u(0)=0$ and $u(a)=0$ with $\sin a \neq 0$
(a) Show that

$$u(x)=\int_0^x f(y)\sin(x-y)dy+c\,\sin x$$

where c is such that

$$c\,\sin a=-\int_0^a f(y)\sin(a-y)dy$$

Hence

$$u(x)=\frac{1}{\sin a}\left[\int_0^x f(y)\sin(x-y)\,\sin a dy+\int_0^a f(y)\sin(a-y)\sin x\,dy\right]$$

(b) Prove that

$$G(x,y)=\begin{cases}\frac{\sin y\sin(x-a)}{\sin a} & y\leq x\\ \frac{\sin(y-a)\sin x}{\sin a} & x\leq y\end{cases}$$

when $\sin a\neq 0$
(c) If $\sin a=0$, show that the above equations has no solution unless $f(x)$ satisfies the condition

$$\int_0^a f(x)\sin(a-x)dx=0$$

in which case there are infinitely many solutions , each of the form

$$u(x)=\int_0^x f(y)\,\sin(x-y)dy+C\sin x$$

where C is an arbitrary constant

problem 1

b) solution of the homogeneous eqn.

$$u_1 = \cos x , \quad u_2 = \sin x \qquad , \quad x \in (0, a)$$

$$G(x,y) = \begin{cases} a_1 \cos x + b_1 \sin x , & x \le y \\ a_2 \cos x + b_2 \sin x , & x \ge y \end{cases}$$

BCs : at $x = 0$ $\qquad a_1 = 0$

at $x = a$ $\qquad a_2 \cos a + b_2 \sin a = 0$

$$b_2 = -\frac{\cos a}{\sin a} a_2 , \qquad \sin a \neq 0$$

Continuity: at $x = y$

$$b_1 \sin y = a_2 \cos y + b_2 \sin y$$

$$= a_2 \cos y - \frac{\cos a}{\sin a} a_2 \sin y$$

$$= \frac{a_2}{\sin a} \sin(a-y)$$

$$b_1 \sin y = \frac{a_2}{\sin a} \sin(a-y)$$

$$- a_2 \sin y + b_2 \cos y - b_1 \cos y = 1$$

$$\cancel{a_2 \sin y = a_2 + \frac{\cos a}{\sin a} \cos y}$$

$b_1 = \rho \sin(a-y)$

$a_2 = \rho \sin a \sin y$

$$-\rho \sin a \sin^2 y - \frac{\cos a}{\sin a} \rho \sin a \sin y \cos y$$
$$- \rho \sin(a-y) \cos y = 1$$

$$-\rho \sin a \sin^2 y - \rho \cos a \cancel{\sin y} \cos y$$
$$-\rho \left(\sin a \cos^2 y - \cos a \cancel{\cos y \sin y} \right) = 1$$

$$\Rightarrow -\rho \sin a = 1 \quad \Rightarrow \rho = -1/\sin a$$

$$G(x,y) = \begin{cases} \dfrac{\sin(y-a)\sin x}{\sin a}, & x \le y \\[2ex] \dfrac{\sin(x-a)\sin y}{\sin a}, & x \geq y \end{cases}$$

a) Then

$$u(x) = \int_0^a G(x,y) f(y)\, dy = \int_0^x G(x,y) f(y)\, dy + \int_x^a G(x,y) f(y)\, dy$$

$$= \int_0^x \frac{\sin(x-a)}{\sin a} \sin y \, f(y)\, dy + \int_x^a \frac{\sin x}{\sin a} \sin(y-a) f(y)\, dy$$

$$u(x) = \int_0^x \frac{\sin(x-a)}{\sin a} \sin y \, f(y)\,dy$$

$$+ \int_0^x \frac{\sin x}{\sin a} \sin(y-a) \, f(y)\,dy$$

$$+ \int_0^a \frac{\sin x}{\sin a} \sin(y-a) \, f(y)\,dy$$

$$= \int_0^x \frac{\sin(x-a)\sin y - \sin x \sin(y-a)}{\sin a} f(y)\,dy$$

$$+ \frac{\sin x}{\sin a} \int_0^a \sin(y-a) \, f(y)\,dy$$

$$= \frac{1}{\sin a} \int_0^x \Big[\sin y \left(\cancel{\sin x \cos a} - \sin a \cos x \right) - \sin x \left(\cancel{\sin y \cos a} - \sin a \cos y \right) \Big] f(y)\,dy$$

$$+ \frac{\sin x}{\sin a} C$$

$$= \frac{1}{\sin a} \left[\int_0^x \sin a \sin(x-y) \, f(y)\,dy + C \sin x \right]$$

$$C = \int_0^a \sin(y-a) \, f(y)\,dy$$

$$u(x) = \frac{1}{\sin a}\left[\sin a \int_0^x \sin(x-y) f(y)\, dy + c \sin x\right]$$

$$= \int_0^x \sin(x-y) f(y)\, dy + \frac{c}{\sin a} \sin x$$

c) If $\sin a = 0 \Rightarrow c = 0$

or $\int_0^a \sin(y-a) f(y)\, dy = 0$

$\frac{c}{\sin a} \to \alpha$ a constat

$$u(x) = \int_0^x \sin(x-y) f(y)\, dy + \alpha \sin x$$

if $\int_0^a \sin(y-a) f(y)\, dy = 0$ there are infinitely many solution if $\neq 0$ there exist no solution

Problem 2 : Solved in Class. See also Lecture Notes

Problem 3

$$2zu'' + (1-2z)u' - u = 0$$

a) $2z(u'-u)' + u' - u = 0$

let $u' - u = y$

$$2zy' - y = 0 \Rightarrow \frac{y'}{y} - \frac{1}{2z} = 0$$

$$\frac{y}{\sqrt{z}} = c_1 \quad \text{constant}$$

$$u' - u = c\sqrt{z} \rightarrow (e^{-z}u)' = c_1\sqrt{z}e^{-z}$$

$$e^{-z}u(z) = c_1\int^{z}\sqrt{z'}\,e^{-z'}dz' + c_2$$

$$u(z) = c_1 e^{z}\int^{z}\sqrt{z'}\,e^{-z'}dz' + c_2 e^{z}$$

b) using the Frobenius method:

$$p(z) = \frac{1-2z}{2z} \rightarrow z\,p(z) = \frac{1}{2} - z$$

$$a_0 = 1/2 \quad , \quad q(z) = -\frac{1}{2z} \rightarrow z^2 q(z) = -\frac{z}{2}$$

$$b_0 = 0$$

Index equation: $r(r-1) + a_0 r + b_0 = r^2 - r + \frac{1}{2}r$

$$= r(r - 1/2) = 0$$

$r_1 = 1/2$, $r_2 = 0$ $\quad r_1 - r_2 = 1/2$ not an integer

Hence both solutions do not have logarithmic singularity.

$$z = \sum_{n=0}^{\infty} c_n z^{n+r}$$

$$2 \sum_{n=0} (n+r)(n+r-1) c_n z^{n+r-1}$$

$$+ \sum_{n=0} c_n (n+r) z^{n+r-1} - 2 \sum_{n=0} (n+r) c_n z^{n+r}$$

$$- \sum c_n z^{n+r} = 0$$

$$= \sum_{n=0}^{\infty} \left[2(n+r)(n+r-1) + (n+r) \right] c_n z^{n+r-1}$$

$$- \sum_{n=1}^{\infty} \left[2(n+r) + 1 \right] c_n z^{n+r} = 0$$

$$= (2r(r-1) + r) c_0 z^{r-1} + \sum_{n=1}^{\infty} \left[(n+r)(2n+2r-1) c_n z^{n+r-1} \right.$$

$$- \sum_{n=0}^{\infty} \left[2n + 2r + 1 \right] c_n z^{n+r} = 0$$

$$= 2r(r - 1/2) c_0 z^{r-1} + \sum_{n=1}^{\infty} \left[(n+r)(2n+2r-1) c_n z^{n+r-1} \right.$$

$$- \sum_{n=0}^{\infty} (2n+2r+1) c_n z^{n+r} = 0$$

$$= 2r(r-1/2)c_0 z^{r-1} + \sum_{n=0}^{\infty} (n+r+1)(2n+2r+1)c_{n+1} z^{n+r}$$

$$- \sum_{n=0} (2n+2r+1)c_n z^{n+r} = 0$$

$$\Rightarrow \quad 2r(r-1/2) = 0 \qquad \text{and}$$

$$(2n+2r+1)\left[(n+r+1)c_{n+1} - c_n\right] = 0 \qquad n \geq 0$$

$$c_{n+1} = \frac{c_n}{n+r+1} \qquad n \geq 1$$

$$r = 1/2 \qquad c_{n+1} = \frac{c_n}{n+3/2} = \frac{2c_n}{2n+3}$$

$$n=0 \qquad c_1 = \frac{2c_0}{3}$$

$$n=1 \qquad c_2 = \frac{2c_1}{5} = \frac{2^2 c_0}{3\cdot 5}$$

$$n=2 \qquad c_3 = \frac{2c_2}{7} = \frac{2^3 c_0}{3\cdot 5\cdot 7}$$

$$\vdots$$

$$n \qquad c_n = \frac{2^n c_0}{1\cdot 3\cdot 5\cdots(2n+1)}$$

$$u_1(z) = z^{1/2} \sum_{n=0}^{\infty} \frac{(2z)^n}{1\cdot 3\cdot 5\cdots(2n+1)}$$

$r = 0$

$$c_{n+1} = \frac{c_n}{n+1}$$

$n=0 \qquad c_1 = \frac{c_0}{1}$

$n=1 \qquad c_2 = \frac{c_1}{2} = \frac{c_0}{1\cdot 2}$

$n=2 \qquad c_3 = \frac{c_2}{3} = \frac{c_0}{1\cdot 2\cdot 3}$

$\vdots$

$n \qquad c_n = \frac{c_0}{n!}$

$$u_2(z) = \sum_{n=0} \frac{z^n}{n!} = e^z$$

$$\Rightarrow \quad u(z) = \alpha\, e^z + \beta\, z^{1/2} \sum_{n=0}^{\infty} \frac{(2z)^n}{1\cdot 3 \cdots (2n+1)}$$

α and β are constant.

problem 4 : Solved in class. See also lecture Notes.