SOLUTIONS

MATH 543

METHODS OF APPLIED MATHEMATICS I
First Midterm Exam

November 08, 2019, Friday 13.40-15.30, SA-Z19

PROBLEMS: *There are four questions in this exam. Solve as many problems as you wish. The total number of points must be at least 100.*

[50]**1.** Let

$$C_n(x) = \frac{1}{w(x)} \frac{d^n}{dx^n} (w(x) s(x)^n), \quad n = 0, 1, 2, \cdots, x \in [a, b]$$

where
(a) $w(x) > 0$ for all $x \in [a, b]$.
(b) $s(x)$ is a polynomial of degree ≤ 2 with real distinct roots.
(c) $C_1(x)$ is a polynomial of degree 1.
(d) $w(a)s(a) = w(b)s(b) = 0$ (boundary conditions).
a. Prove that C_n's are polynomials of degree n.
b. Prove that C_n's are orthogonal to any polynomial of degree less then n.(Orthogonality with respect to the $L^2_w(a, b)$ inner product).
c. Show that C_n's satisfy a second order linear differential equation.

2[20]. $C_{(n)}(x)$'s are the classical orthogonal polynomials of degree n, $F_N(x)$ and $Q_N(x)$ are defined as

$$F_N(x) = \sum_{n=1}^{N} a_n \, C_{n+r}(x) \, C_n(x), \tag{1}$$

$$Q_N(x) = \sum_{n=k}^{N} b_n \, C_{n-k}(x) \, C_{n+k}(x) \, C_{2n+r}(x), \tag{2}$$

where a_n and b_n are real constants, and $r = 1, 2, \cdots$ and $k = 0, 1, 2, \cdots$.

(a) Prove that $\int_a^b w(x) \, Q_N(x) \, p_\ell(x) dx = 0$ for all $\ell < r$ and for all $k = 0, 1, 2, \cdots$. Here $p_\ell(x)$ is a polynomial of degree $\leq \ell$.

(b) Show that $F_N(x)$ and $Q_N(x)$ have (r-1) real roots in $[a, b]$.

3[30]. An ordinary differential equation is given as

$$xy'' + (x+1)y' + y = x.$$

One of the solutions of the associated homogeneous equation is given $y_1(x) = e^{-x}$. Find the general solution of this differential equation.

4[30]. 5. Consider the following sequence:

$$h_n(x) = \begin{cases} 0 & \text{if } x \leq \frac{-1}{n} \\ \frac{(nx+1)}{2} & \text{if } \frac{-1}{n} \leq x \leq \frac{1}{n} \\ 1 & \text{if } x \geq \frac{1}{n} \end{cases}.$$

(i) Prove that $h_n(x) \to \theta(x)$ where $\theta(x)$ is the step function and
(ii) $\frac{dh_n(x)}{dx} \to \delta(x)$. Hence formally we may write that $\frac{d\theta(x)}{dx} = \delta(x)$

Pr 1: See Lecture 4 (page 48)

Pr. 2.

a) $\int_a^b w(x)\, Q_N(x)\, P_e(x)\, dx$

$= \sum b_n \int_a^b w(x)\, C_{2n+r}(x)\, C_{n-k}(x)\, C_{n+k}(x)\, P_e(x)\, dx$

The degree of the product $C_{n+k} \cdot C_{n-k} \cdot P_e(x)$ is equal to $2n+l$. There we can let

$C_{n+k}(x)\, C_{n-k}(x)\, P_e(x) = P_m(x)$, $m = 2n+l$.

if $l < r$ then

$$\int_a^b w(x)\, C_{2n+r}(x)\, P_m(x)\, dx = 0$$

becaue $m < 2n+r$ for all N.

b) Both F_N and Q_N satisfy

$$\int_a^b w(x)\, F_N(x)\, dx = \int_a^b w(x)\, Q_N(x)\, dx = 0$$

Since $w(x) > 0$ for all $x \in (a,b)$. Then both $F_N(x)$ and $Q_N(x)$ have at least on root in $[a,b]$. Let us assume that, for instance Q_N has m-number of roots in (a,b)

i.e $\quad Q_N(x) = \prod_{i=1}^{m} (x - x_i)\, h(x)$

where $h(x)$, without loosing any generality, is positive definite in $[a,b]$. Then

$$\int_a^b w \prod_{i=1}^{M} (x-x_i)\, Q_N(x)\, dx$$

$$= \int_a^b w \left(\prod_{i=1}^{M} (x-x_i) \right)^2 h(x)\, dx > 0$$

On the other hand

$$\int_a^b w \prod_{i=1}^{M}() \, Q_N(x)\, dx$$

$$= \int w \sum_{n=k}^{N} b_n \prod_{i=1}^{M} (x-x_i)\, C_{n-k}(x)\, C_{n+k}(x)\, C_{2n+r}(x)\, dx$$

$$= \sum_{n=k}^{N} b_n \int_a^b w\, C_{2n+r}(x)\, q_{2n+M}(x)\, dx$$

where $q_{2n+M}(x) = \prod_{i=1}^{M} (x-x_i)\, C_{n-k}(x)\, C_{n+k}(x)$

and since $M < r$. then

$$\int_a^b w \prod_{i=1}^{M} (x-x_i)\, Q_N(x)\, dx = 0$$

We get a contradiction. To resolve the contradiction we must let $M = r$. Hence Q_N has r number of real roots in $[a,b]$

Pr. 3

$$xy'' + (x+1)y' + y = x$$

with $y_1(x) = e^{-x}$.

a) let $y_2(x) = h(x)e^{-x}$

$$y_2' = h'e^{-x} - h\cdot e^{-x}$$

$$y_2'' = h''e^{-x} - 2h'e^{-x} + he^{-x}$$

$$xy_2'' + (x+1)y_2' + y_2 = xh''e^{-x} - 2xh'e^{-x} + \cancel{hxe^{-x}}$$
$$+ (x+1)h'e^{-x} - (\cancel{x}+1)he^{-1} + he^{-x} = 0$$

$$\Rightarrow \quad xh'' - 2xh' + (x+1)h' = 0$$

$$xh'' + (1-x)h' = 0$$

$$xh'' + h' - xh' = 0$$

$$(xh')' - (xh') = 0$$

$$\Rightarrow (xh'e^{-x})' = 0$$

$$\Rightarrow xh'e^{-x} = \beta \quad \text{(constant)}$$

$$h' = \beta\frac{e^x}{x} \quad \Rightarrow h(x) = \beta\int^x \frac{e^\xi}{\xi}d\xi.$$

$$y_2(x) = e^{-x}\int^x \frac{e^\xi}{\xi}d\xi$$

b) Particular solution

$$y_p = g\, e^{-x}$$

$$y_p'' + (x+1)\, y_p' + y_p = x$$

$$\text{or } x g'' - 2x g' + (x+1) g' = x e^{x}$$

$$(x g')' - (x g)' = x e^{x}$$

$$(x g' e^{-x})' e^{x} = x \cdot e^{x}$$

$$\cancel{x g' e^{-x} = \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x}}$$

$$x g' e^{-x} = \frac{x^2}{2}$$

$$g' = \frac{x}{2} e^{x} \Rightarrow g(x) = \frac{1}{2}(x-1) e^{x}$$

$$\Rightarrow y_p = \frac{1}{2}(x-1).$$

The general solution is

$$y(x) = \alpha e^{-x} + \beta e^{-x} \int^{x} \frac{e^{\xi}}{\xi}\, d\xi + \frac{1}{2}(x-1)$$

where α and β are arbitrary constants.

pr.5/ See lecture Note 6 on Generalized Functions. This problem is solved also in class.