MATH 543

METHODS OF APPLIED MATHEMATICS I

Final Makeup Exam

SOLUTIONS

January 18, 2019

Friday 09.30-11.30, SAZ-18

PROBLEMS: There are five problems. Choose any four of them.

[30] **1**. Consider a second order linear differential equation $L_x\, u = f(x), x \in (a, b)$ with homogeneous boundary conditions $B_1(u) = 0,\ B_2(u) = 0$. Provided the homogeneous equation $L_x\, u = 0$ has no nontrivial solutions satisfying the boundary conditions $B_1(u) = 0,\ B_2(u) = 0$, **show that** the Green's function associated with above boundary value problem exists and is unique. **Then** the solution of the boundary value problem given by

$$u(x) = \int_a^b w(y)\, G(x, y)\, f(y)\, dy$$

is unique.

[30]**2**. Let

$$u'' + p(z)\, u' + q(z)\, u = 0$$

be a differential equation having a regular singular point at $z = z_0$ so that the difference of the indices is an integer, $r_1 - r_2 = N$. **Show that** the solution corresponding to the second index r_2 has the form

$$u_2(z) = C\, u_1(z)\, \ln(z - z_0) + (z - z_0)^{r_2} \sum_{n=0}^{\infty} d_n\, (z - z_0)^n$$

where C is a constant.

[30]**3**. **Prove that** a second order linear differential equation having only two regular singular points is of Euler type (a differential equation which can be transformable to a differential equation with constant coefficients).

[30]**4**. Let $\epsilon y'' + 2y' + y(1 + \epsilon y) = 0$ for $0 < \epsilon \ll 1,\ t > 0$ with $y(0) = 0$, $y(1) = 1$. **Obtain a uniformly valid** approximation of this boundary value problem.

[30]5. (a) **Using the WKB approximation find an approximation** for the large eigenvalues and corresponding eigenfunctions of $y'' + \lambda e^{4x}\, y = 0$ for $0 < x < 1$, $\lambda >> 1$, with $y(0) = y(1) = 0$.

(b) **Find a two term** approximation (first two terms) for large λ of the integral

$$\int_0^{\pi/2} e^{-\lambda \tan^2 \theta}\, d\theta$$

SOLUTION

1) See the lecture notes 8

2) See the lecture notes 9

3) See the lecture note 10

4) $$\varepsilon y'' + 2y' + y(1+\varepsilon y) = 0, \quad 0<\varepsilon \ll 1, \quad t>0$$

$$y(0) = 0, \quad y(1) = 1.$$

This is a boundary value problem which can be solved by the singular perturbation method.

<u>outer region</u>: we can use RPM, then zeroth order perturbation gives

$$2y_0' + y_0 = 0, \quad y_0(1) = 1.$$

$$\Rightarrow y_0(t) = e^{\frac{1}{2} - \frac{1}{2}t}$$

<u>Inner region</u>: let $t = \delta\tau$ then

$$\frac{\varepsilon}{\delta^2} y_{\tau\tau} + \frac{2}{\delta} y_\tau + y(1+\varepsilon y) = 0$$

The best choice is $\delta = \varepsilon \Rightarrow$

$$y_{in\,\tau\tau} + 2y_{in\,\tau} + \varepsilon y_{in}(1+\varepsilon y_{in}) = 0$$

$$\Rightarrow y_{in}(\tau) = A + Be^{-2\tau} \qquad y(0) = 0$$

$$\Rightarrow B = -A$$

$$y_{in}(\tau) = A - Ae^{-2t/\varepsilon}$$

Matching region: let $t = \sqrt{\varepsilon}\,\eta$ and let

$\varepsilon \to 0$

$$\lim_{\varepsilon \to 0} y_0(\sqrt{\varepsilon}\,\eta) = \lim_{\varepsilon \to 0} y_{in}(\sqrt{\varepsilon}\,\eta)$$

$$e^{1/2} = A$$

$$\Rightarrow \quad y_{in}(t) = e^{1/2} - e^{\frac{1}{2} - 2t/\varepsilon}$$

Approximate soln.

$$y_{ap} = y_0 + y_{in} - y_{match} = e^{\frac{1}{2} - \frac{1}{2}t} - e^{\frac{1}{2} - 2t/\varepsilon}$$

Uniform validity of the soln

$$r(t, \varepsilon) = \varepsilon y_{ap}'' + 2 y_{ap}' + y_{ap}(1 + \varepsilon y_{ap})$$

$$= \frac{\varepsilon}{4} e^{\frac{1}{2} - \frac{1}{2}t} - e^{\frac{1}{2} - 2t/\varepsilon} + \varepsilon^2 \left(e^{\frac{1}{2} - \frac{1}{2}t} - e^{\frac{1}{2} - 2t/\varepsilon} \right)$$

$$\lim_{\varepsilon \to 0} r(t, \varepsilon) = 0 \qquad \text{uniformly on } [0, \infty).$$

SOLUTIONS

5. a) $y'' + \lambda e^{4x} y = 0 \qquad 0<x<1, \quad \lambda \gg 1$

Lecture notes 12 (page 37)

$$y(x) = \frac{c_1}{\sqrt{k}} \cos\left(\sqrt{\lambda}\int_0^x \sqrt{k(x)}\,dx\right) + \frac{c_2}{\sqrt{k}} \sin\left(\sqrt{\lambda}\int_0^x k(x)dx\right)$$

where $k = e^{2x}$

$y(0) = 0 \quad \Rightarrow c_1 = 0$

$y(1) = 0 \quad \Rightarrow \sqrt{\lambda}\int_0^1 e^{2x}dx = \cancel{\ldots} = n\pi, \quad n=1,2\ldots$

$$\lambda_n = \frac{4n^2\pi^2}{(e^2-1)} \qquad \underline{\text{eigen values}} \quad \text{for } n \gg 1.$$

<u>eigen functions</u>

$$y_n(x) = c_1 e^{-x} \sin\left[\sqrt{\lambda_n}\left(\tfrac{1}{2}e^{2x} - \tfrac{1}{2}\right)\right], \quad n=1,2,\ldots$$

c_1 is the normalization constant.

b) $$I = \int_0^{\pi/2} e^{-\lambda \tan^2\theta}\, d\theta = ? \qquad \text{for } \lambda \gg 1$$

let $\tan\theta = u \qquad d\theta \sec^2\theta = du$

or $d\theta = \dfrac{du}{1+u^2}$

$$\Rightarrow I(\lambda) = \int_0^\infty e^{-\lambda u^2}\frac{du}{1+u^2} = \int_0^a e^{-\lambda u^2}\left[1 - u^2 + u^4 + \ldots\right]du \Rightarrow$$

Since $\displaystyle\int_0^\infty e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$ $\Rightarrow$ $\underline{I(\lambda) = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}} - \frac{1}{4}\sqrt{\frac{\pi}{\lambda^3}}} + O(\lambda^{5/2})$

$a > 0$