MATH 543

METHODS OF APPLIED MATHEMATICS I
Second Midterm Exam SOLUTIONS

December 6, 2018
Thursday 13.40-15.30
SA-Z02

PROBLEMS: There are four questions in this exam. Show all your work.

1[30]. Solve the following boundary value problem by using Green's function method

$$\frac{d^2u}{dx^2} = f(x), \quad x \in (0,a),$$
$$u(0) - \alpha\, u'(0) = \mu, \quad u(a) - \beta\, u'(a) = \nu$$

where α, β, μ and ν are some given real numbers.
(a) When $\beta - \alpha \neq a$,
(b) When $\beta - \alpha = a$ and $\mu = \nu = 0$

2[20]. Let $u'' - 2x\,u' + 2\,u = e^{x^2}$. One of the solutions of the homogenous equation is $u_1 = x$. Using the method of variation of constants find the general solution of this equation.

3[25]. Let $u'' + p(z)u' + q(z)u = 0$ where $p(z)$ and $q(z)$ are analytic function at all points of a region D except some finite number points and the boundary conditions be $u(z_0) = A$, $u'(z_0) = B$ where $z = z_0$ is an ordinary point of the differential equation. Prove that the solution of this differential equation about the ordinary point $z = z_0$ exists and unique.

4[25]. Prove that the Green's function of the following boundary value problem is symmetric.

$$L(u) = f(x), \quad x \in (a,b)$$
$$B_1(u) = B_2(u) = 0$$

where L is a real and self adjoint operator.

1) $u'' = f(x), \quad x \in (0, a)$.

$u(0) - \alpha u'(0) = \mu$

$u(a) - \beta u'(a) = \nu$

Solution: First let us check whether there exists a solution of the homogenous equation satisfying both boundary condition

Most general soln. of the homogenous eqn is $v(x) = a_1 + b_1 x$ where a_1 and b_1 are constants.

$B_1(v) = a_1 - \alpha b_1 = 0$

$B_2(v) = a_1 + b_1 a - \beta b_1 = 0$

$\Rightarrow \quad b_1 a - \beta b_1 + \alpha b_1 = 0$

$b_1 (\alpha - \beta + a) = 0$

if $b_1 = 0 \Rightarrow a_1 = 0 \Rightarrow$ trivial soln.

so $b_1 \neq 0 \Rightarrow \alpha - \beta + a = 0$

Hence if $\beta - \alpha = a$ then

There is a solution of the homogenous equation satisfying both BC. Hence the standard GF method does not work.

a) If $\beta - \alpha \neq a$. There is no~~t~~ nontrivial solution of the homogeneous equation satisfying both BC. Hence we can use the standard GF method.

$$G(x,y) = \begin{cases} A + Bx, & x \leq y \\ C + Dx, & x \geq y \end{cases}$$

- We find $G(x,y)$ as if the BCs are homogeneous.

i) $A + By = C + Dy \Rightarrow A - C = (D - B)y$

ii) $D - B = 1 \Rightarrow A - C = y$

iii) $A - \alpha B = 0 \Rightarrow A = \alpha B$

iv) $C + Da - \beta D = 0 \Rightarrow C = (\beta - a) D$

$$\left.\begin{aligned} A &= (\beta - a) D + y \\ B &= D - 1 \\ C &= (\beta - a) D \end{aligned}\right\} \quad \begin{aligned} (\beta - a) D + y &= \alpha D - \alpha \\ (\alpha - \beta + a) D &= \alpha + y \end{aligned}$$

$$\Rightarrow \quad D = \frac{\alpha + y}{\alpha - \beta + a}, \quad C = \frac{(\alpha + y)(\beta - a)}{\alpha - \beta + a}$$

$$B = \frac{\alpha + y}{\alpha - \beta + a} - 1 = \frac{\cancel{+\alpha} + y - \cancel{\alpha} + \beta - a}{\alpha - \beta + a} = \frac{\beta + y - a}{\alpha - \beta + a}$$

$$A = \frac{(\beta - a)(\alpha + y)}{\alpha - \beta + a} + y$$

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$$A = \frac{(\beta - a)(\alpha + y) - (\beta - a)y + \alpha y}{\alpha - \beta + a} = \frac{(\beta - a)\alpha + \alpha y}{\alpha - \beta + a}$$

$$A = \frac{(\beta - a + y)\alpha}{\alpha - \beta + a}$$

$$G(x,y) = \begin{cases} \dfrac{\alpha(\beta - a + y)}{\delta} + \dfrac{\beta + y - a}{\delta} x = \dfrac{(\beta - a + y)(\alpha + x)}{\delta}, & x \le y \\[2ex] \dfrac{(\alpha + y)(\beta - a)}{\delta} + \dfrac{(\alpha + y)}{\delta} x = \dfrac{(\alpha + y)(\beta - a + x)}{\delta}, & x \geq y \end{cases}$$

$$\delta = \alpha - \beta + a \neq 0$$

Solution with homogenous BCs

$$u_0(x) = \int_0^a G(x,y) f(y)\, dy = \int_0^x G_{x \geq y} f\, dy + \int_x^a G_{\le} f\, dy$$

$$= \frac{1}{\delta} \int_0^x (\alpha + y)(\beta - a + x) f(y)\, dy + \frac{1}{\delta} \int_x^a (\beta - a + y)(\alpha + x) f\, dy$$

$$u_0(x) = \frac{1}{\delta}(\beta - a + x) \int_0^x (\alpha + y) f(y)\, dy + \frac{1}{\delta}(\alpha + x) \int_x^a (\beta - a + y) f\, dy$$

- To find the solution of the inhomogeneous BV problem we let

$$u(x) = Ax + B + u_0(x)$$

we find the constants from the inhomogeneous BCs:

$$u(0) - \alpha u'(0) = B - \alpha A = \mu \Rightarrow B = \mu + \alpha A.$$

$$u(a) - \beta u'(a) = Aa + B - \beta A = \nu \Rightarrow Aa + \mu + \alpha A - \beta A = \nu$$

$$\Rightarrow A(a + \alpha - \beta) = \nu - \mu \quad \Rightarrow A = \frac{-\mu + \nu}{\delta}$$

$$B = \mu + \frac{\alpha(\nu - \mu)}{\delta} = \frac{\mu(a + \cancel{\alpha} - \beta) + \alpha\nu - \cancel{\alpha\mu}}{\delta}$$

$$= \frac{\mu(a - \beta) + \alpha\nu}{\delta}$$

$$\Rightarrow u(x) = \frac{\nu - \mu}{\delta} x + \frac{\mu(a - \beta) + \alpha\nu}{\delta}$$

$$+ \frac{1}{\delta}(\beta - a + x)\int_0^x (\alpha + y) f(y)\, dy$$

$$+ \frac{1}{\delta}(\alpha + x)\int_x^a (\beta - a + y) f(y)\, dy$$

b) When $\beta - \alpha = a$ or $\delta = 0$ there exists a nontrivial solution of the homogenous equation, $v = x + \alpha$ satisfying both BCs. Then we use generalized GF

$$G_{xx}(x,y) = \delta(x-y) - \frac{1}{\Delta}(x+\alpha)(y+\alpha).$$

where $\Delta = \frac{1}{3}[a^3 + 3a^2\alpha + 3a\alpha^2]$. Then for $x \neq y$

$$G_{xx} = -\frac{1}{\Delta}(x+\alpha)(y+\alpha).$$

This leads to

$$G(x,y) = \begin{cases} -\frac{1}{6\Delta}(x+\alpha)^3(y+\alpha) + A(x+\alpha) + B, & x \le y \\ -\frac{1}{6\Delta}(x+\alpha)^3(y+\alpha) + C(x+\alpha) + D, & x \geq y \end{cases}$$

$B_1(G) = 0 \Rightarrow \quad B = -\frac{1}{3\Delta}\alpha^3(y+\alpha)$

$B_2(G) = 0 \Rightarrow \quad D = -\frac{1}{3\Delta}(a+\alpha)^3(y+\alpha)$

continuity at $x = y \Rightarrow \quad A - C = -1.$

Jump condit. $\Rightarrow \quad C - A = 1$

one parameter left arbitrary

constraint: $\int_0^a (y+\alpha) f(y)\, dy = 0$

if $\int_0^a (y+\alpha) f(y)\,dy \neq 0$ there exist no solution.

If $\int_0^a (y+\alpha) f(y)\,dy = 0$ there are infinitely many solution:

$$u(x) = \gamma\,(x+\alpha) + (x+\alpha)\int_0^x f(y)\,dy - \int_0^x (y+\alpha) f(y)\,dy$$

where γ is an arbitrary constant.

2) using the method of variation of constants

a) let $u_2 = x\, h(x)$, then

$$(xh)'' - 2x(xh)' + 2xh = 0$$

$$(xh'' + 2h') - 2x(xh' + \cancel{h}) + 2\cancel{x}h = 0$$

$$x h'' + 2(1 - x^2) h'$$

$$\frac{h''}{h'} = -2\left(\frac{1}{x} - x\right) \Rightarrow \ln h' = -2\ln x + x^2$$

$$h' = \frac{1}{x^2} e^{x^2} \Rightarrow h(x) = \int^{x} \frac{e^{x'^2}}{x'^2} dx'$$

$$u_2(x) = x \int^{x} \frac{e^{x'^2} dx'}{x'^2}$$

b) $u_p = x\, g(x)$

$$(xg)'' - 2x(xg)' + 2xg = e^{x^2}$$

$$x g'' + 2(1 - x^2) g' = e^{x^2}$$

$$g'' + 2\left(\frac{1}{x} - x\right) g' = \frac{1}{x} e^{x^2}$$

from the first part $\frac{h''}{h'} = -2\left(\frac{1}{x} - x\right)$

$$\Rightarrow g'' - \frac{h''}{h'} g' = \frac{1}{x} e^{x^2}$$

$$\left(\frac{g'}{h'}\right)' = \frac{1}{x h'} e^{x^2} = x \Rightarrow \frac{g'}{h'} = \frac{1}{2} x^2$$

$$g'(x) = \frac{1}{2} e^{x^2} \Rightarrow g(x) = \frac{1}{2} \int^{x} e^{x'^2} dx'$$

$$\Rightarrow \quad v_p(x) = \frac{x}{2}\int^{x} e^{x'^2}\,dx'$$

Then the most general solution is

$$u(x) = Ax + Bx\int^{x}\frac{1}{x'^2}e^{x'^2}\,dx' + \frac{x}{2}\int^{x} e^{x'^2}\,dx'$$

3) See Lecture Notes

4) See Lecture Notes.