MATH 543

METHODS OF APPLIED MATHEMATICS I
First Midterm Exam

November 01, 2018
Thursday 13.40-15.30
SA-Z02

SOLUTIONS

PROBLEMS: There are four questions in this exam.

1[25]. Let $u_m(x)$, $(m = 0, 1, 2, \cdots)$ be one of the classical orthonormal polynomials with weight function $w(x)$ and $x \in [a, b]$. Then the sequence

$$h_n(x) = w(x) \sum_{k=0}^{n} u_k(x)\, u_k(x_0)$$

where $x_0 \in [a, b]$ defines the delta function $\delta(x - x_0)$
(b) Let $u_n(x) = \frac{1}{\sqrt{2\pi}}\, e^{-inx}$, $(n = 0, 1, 2, \cdots)$ with $x \in [-\pi, \pi]$. Prove that the sequence

$$g_n(x) = \sum_{k=0}^{n} u_k(x)\, u_k(x_0)$$

where $x_0 \in [-\pi, \pi]$ defines also the delta function $\delta(x - x_0)$

2[25]. Assume that there exists an orthonormal basis $|e_i >$, $(i = 1, 2, \cdots)$ in $L^2_w(a, b)$. Then , for any $|f > \in L^2_{a,b}$, the sequence of vectors

$$|f_k >= \sum_{i=1}^{k} f^i\, |e_i >$$

with

$$f^i =< e_i | f >$$

has $|f >$ as the limit vector in the sense that

$$\lim_{k \to \infty} \rho(|f >, |f_k >) = 0$$

3[25]. If $(e_1, e_2, \cdots, e_n)$ is a finite orthonormal family of functions in $L^2_w(a,b)$ and $f \in L^2_w(a,b)$, then

$$||f - \sum_{k=1}^{n} f^k \, |e_k> ||$$

has its minimum value when $f^k =< f|e_k >$ and

$$\sum_{k=1}^{n} | < f|e_k > |^2 \leq ||f||^2$$

4[25]. $C_{(n)}(x)$'s are the classical orthogonal polynomials of degree n and k_n, k'_n, h_n and $F_N(x)$ are defined as

$$C_{(n)}(x) = k_n \, x^n + k'_n \, x^{n-1} + p_{(n-2)}(x), \tag{1}$$

$$\int_a^b w(x)(C_{(n)}(x))^2 dx = h_n \tag{2}$$

$$F_N(x) = \sum_{n=1}^{N} a_n \, C_{n-1}(x) \, C_n(x) \tag{3}$$

where $p_{(k)}(x)$ is a polynomial of degree less or equal to k and a_n are real constants.

(a) Show that $F_N(x)$ has at least one real root in $[a,b]$.

(b) Find α_n

$$k_n \, C_{(n+1)}(x) - k_{n+1} \, x \, C_{(n)}(x) = \alpha_n x^n + p_{(n-1)}(x)$$

(c) Evaluate the integral $\int_a^b w(x)x \, (C_{(n)})^2 \, dx$

Solution of the first Midterm Exam.

1) a) $$h_n(x) = \omega(x)\sum_{k=0}^{n} u_k(x)\, u_k(x_0), \qquad x, x_0 \in [a,b]$$

$$\lim_{n\to\infty} \int_{a}^{b} h_n(x)\, g(x)\, dx$$

$$= \lim_{n\to\infty} \int_0^b \omega(x) \sum_{k=0}^{n} u_n(x)\, u_n(x_0)\, g(x)\, dx$$

$$= \sum_{n\to\infty}^{\infty} \left(\int_0^b \omega(x)\, u_n(x)\, g(x)\,dx \right) u_n(x_0)$$

$$= \sum_{n=0}^{\infty} a_n\, u_n(x_0)$$

where $$a_n = \int_a^b \omega(x)\, u_n(x)\, g(x)\, dx$$

and $$g(x) = \sum_{n=0}^{\infty} a_n\, u_n(x)$$

$$\Rightarrow \lim_{n\to\infty} \int_0^b h_n(x)\, g(x)\, dx = g(x_0)$$

$$= \int_0^b \delta(x - x_0)\, g(x)\, dx$$

for any good function $g(x)$

$$\Rightarrow \quad h_n(x) \to \delta(x - x_0)$$

b) we have to prove that $u_n = \frac{1}{\sqrt{2\pi}} e^{-inx}$

is orthonormal in $[-\pi, \pi]$ and $\omega = 1$.

you can prove that $\langle u_n, u_m \rangle = \delta_{mn}$.

Then $u_n(x)$ form a basis of $L^2[-\pi, \pi]$.

The rest of the proof is exactly the part (a).

2) Please see Lecture Note 2 page 35-36

3) let $\| f - \sum_{k=1}^{n} f^k | e_k \rangle \|$ be a function

where f^k's are unknowns. Then

$$\| f - \sum_{k=1}^{n} f^k | e_k \rangle \|^2 = \left(\langle f | - \sum f^k \langle e_k | \right) \left(| f \rangle - \sum f^i | e_i \rangle \right)$$

$$= \|f\|^2 - \sum_{i=1}^{n} \bar{f^i} \langle f | e_i \rangle - \sum_{i=1}^{n} f^i \langle e_i | f \rangle + \sum_{i=1}^{n} f^i \bar{f^i}$$

$$= \|f\|^2 - \sum_{i=1}^{n} | \langle f | e_i \rangle |^2 + \sum_{i=1}^{n} | f^i - \langle f | e_i \rangle |^2$$

$$\geq \|f\|^2 - \sum_{i=1}^{n} | \langle f | e_i \rangle |^2 \qquad \forall n$$

The minimum value of $\| f - \sum_{i=1}^{n} f^i | e_i \rangle \|$ is attained

when $\sum_{i=1}^{n} | f^i - \langle f | e_i \rangle |^2 = 0$ or $f^i = \langle f | e_i \rangle$

for all $i = 1, 2 ..$

At this minimum value

$$\| f - \sum_{i=1}^{n} f^{k} |e_k\rangle \|^2 = \|f\|^2 - \sum_{i=1}^{n} |\langle f|e_i\rangle|^2$$

hence

$$\|f\|^2 - \sum_{i=1}^{n} |\langle f|e_i\rangle|^2 \geq 0$$

4) a) $\int_a^b w(x)\, F_N(x)\, dx = 0 \quad \forall\, N = 1, 2, \ldots$

hence $F_N(x)$ must change it's sign in $[a,b]$ which means that (IVT) it has a real root in $[a,b]$

b) compare the coefficients of x^n both sides

$$k_n k'_{n+1} - k_{n+1} k'_n = \alpha_n$$

c) use (b), multiply both sides by $w\, C_n$ and integrate in $[a,b]$, we get

$$-k_{n+1} \int_a^b w\, x\, C_n^2\, dx = \alpha_n \int_a^b w\, x^n C_n\, dx$$

$$= \alpha_n \frac{h_n}{k_n}$$

$$\Rightarrow \int_a^b w\, C_n^2\, dx = -\alpha_n \frac{h_n}{k_n \cdot k_{n+1}}$$

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