MATH 543

METHODS OF APPLIED MATHEMATICS I

Second Midterm Exam Solutions

~~March 20~~ April 24, 2017

Monday 10.40-12.30

PROBLEMS: Choose any three of the following four questions:

[35]1. A boundary value problem is given by

$$L(u) = f(x), \quad x \in [a, b], \tag{1}$$

$$B_1(u) = 0, \quad B_2(u) = 0 \tag{2}$$

where ord(L)=2 and L, B_1 and B_2 are linear operators and L is hermitian. Prove that the solution of this boundary value problem exists provided that the solutions of the homogenous equation has no nontrivial solution satisfying both boundary conditions. Here f is a continuous function in $[a, b]$.

Solution: See the Lecture Notes.

[35]2. By using the method of Greens function solve the boundary value problem $u'' = f(x)$ where $x \in [0, 1]$ with $u(0) = \alpha$ and $u'(1) = \beta$ (Here α and β are real constants). Here $f(x)$ is any continuous function in $[0, 1]$.

Solution: A) Find the Greens function as if the BCs are homogenous.

$$G(x,y) = \begin{cases} a_1 x + b_1 & x \le y \\ a_2 x + b_2 & x \ge y \end{cases}$$

a) continuity at $x = y$ ⇒ $(a_1 - a_2) y = b_2 - b_1$

b) Jump cond. at $x = y$ ⇒ $a_2 - a_1 = 1$

d) BC at $x = 0$ ⇒ $b_1 = 0$

e) BC at $x = 1$ ⇒ $a_2 = 0$

$$\Rightarrow G(x,y) = \begin{cases} -x, & x \le y \\ -y & x \ge y \end{cases}$$

B) Now write the soln. as: $u(x) = Ax + B + \int_0^1 G(x,y) f(y)\,dy$

Since $u(0) = \alpha \Rightarrow B = \alpha$, Since $u'(1) = \beta \Rightarrow A = \beta$.

$$\Rightarrow \quad \underline{u(x) = \beta x + \alpha - \int_0^x y f(y)\,dy - x \int_x^1 f(y)\,dy}$$

[35] **3.** Let $J(y) = \int_a^b L(x, y, y')dx$ where $y(a) = y_0$ and $y(b) = y_1$. Here L is a twice differentiable function of its arguments and $y \in C^2[a, b]$. Find the necessary condition for J to have an extremum for a given function y.

Solution: See the Lecture Notes.

[35]4. Let $J(y) = \int_0^1 [(y')^2 + k^2y^2]dx$ with $y(0) = 0$ and $y(1) = 1$. Here k is nonzero real constant. Show that $y_0 = \frac{\sinh(kx)}{\sinh(k)}$ minimizes the functional J.

Solution: $J(y_0 + \varepsilon h) = \int_0^1 [(y_0' + \varepsilon h')^2 + k^2(y_0 + \varepsilon h)^2]dx$

$$\Rightarrow \Delta J = J(y_0 + \varepsilon h) - J(y_0) = 2\varepsilon \int_0^1 (y_0' h' + k^2 y_0 h)\,dx + \varepsilon^2 \int_0^1 (h'^2 + k^2 h^2)\,dx$$

$$y_0' h' + k^2 y_0 h = \frac{k}{\sinh k}(\cosh kx \; h' + k \sinh kx \; h) = \frac{k}{\sinh k}(\cosh kx \; h(x))'$$

$$\int_0^1 (y_0' h' + k^2 y_0 h)\,dx = \frac{k}{\sinh k} \cosh kx \; h(x) \Big|_0^1 = \frac{k}{\sinh k}(\cosh k \; h(1) - h(0)) = 0 \quad \text{Since } h(0) = h(1) = 0$$

$$\Rightarrow J(y_0 + \varepsilon h) - J(y_0) = \varepsilon^2 \int_0^1 (h'^2 + k^2 h^2)\,dx$$

1) $\delta J = \frac{\partial J}{\partial \varepsilon}\Big|_{\varepsilon=0} = 0 \Rightarrow$ Euler-Lagrange eqn. is satisfied meaning that y_0 is a critical point.

2) $\delta^2 J = \frac{\partial^2 J}{\partial \varepsilon^2}\Big|_{\varepsilon=0} = 2\int_0^1 (h'^2 + k^2 h^2)\,dx \geq 0$. Hence at $y = y_0$ J has a local minimum value