MATH 543

METHODS OF APPLIED MATHEMATICS I
Final Exam

May 18, 2017
Thursday 09:00-11.00

PROBLEMS: Choose any four of the following eight questions:

[25]**1**. Let $P_n(x)$ denote the Legendre polynomials of degree n, where $x \in [-1,1]$. The Rodriguez formula for $P_n(x)$ is given by

$$P_n = \frac{(-1)^n}{n!\,2^n}\frac{d^n}{dx^n}(1-x^2)^n$$

for all $n = 0, 1, 2, \cdots$ and $x \in [-1,1]$. First two terms

$$P_n(x) = k_n x^n + k_n' x^{n-1} + \cdots$$

and the norm of P_n

$$h_n = \int_{-1}^{1} P_n(x)^2 dx$$

are given by

$$k_n = \frac{2^n \Gamma(n+1/2)}{n!\,\Gamma(1/2)}, \quad k_n' = 0, \quad h_n = (n+1/2)^{-1}$$

(a) Find the constant α in the following formula

$$(n+1)P_{n+1}(x) = \alpha x P_n(x) - nP_{n-1}(x)$$

(b) Prove that $P_n(x)$ has n real roots in $[-1,1]$.

[25]**2**. Let

$$\frac{1}{w}\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + cu = f, \quad a < x < b,$$

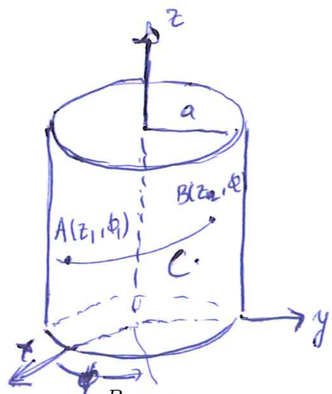
and the boundary conditions $B_1(u) = 0$ and $B_2(u) = 0$ are defined at $x = a$ and $x = b$ respectively. Solve the Green's function and give the solution of the problem. Discuss all possibilities. Here $w(x) > 0$ for all $x \in [a, b]$, p is a differentiable function and f is a continuous function in $[a, b]$

[35] **3**. Let $y'' + y + \varepsilon y^3 = 0$ for $t > 0$ with $y(0) = 1, \;\; y'(0) = 0$. Solve this equation up to second order in ε and justify your approximate solution.

[25]**4**. Using the method of Green's function solve the following boundary value problem

$$u'' + 6u' + 9u = x, \qquad x > 0,$$

and $u(0) = 0$, $u'(0) = 1$.



[35]**5**. Find the curves (geodesics) on a right cylinder so that the distance between any two pints A and B is the shortest on a right cylinder. Any point on the cylinder is defined by the z and ϕ coordinates Here ϕ is the angle measures from the plane $y = 0$. For the cylinder, arc-length is defined as $ds^2 = dz^2 + a^2\, d\phi^2$ where a is the radius of the cylinder. You can assume that Lagrange Function L as $L = (\frac{dz}{ds})^2 + a^2\, (\frac{d\phi}{ds})^2$. Then you minimize the functional $J = \int_A^B L\, ds$.

[25]**6**. **(a)** Prove that for a real Hermitian second order operator, the corresponding Green's function $G(x, y)$ is symmetric, $G(x, y) = G(y, x)$.
(b). Let $u_1(x)$ and $u_2(x)$ be two independent solutions of the homogeneous equation $Lu = 0$, where L is second order linear Hermitian operator. Let $G(x, y)$ be the Green's function of the problem and let the boundary value problem

$$L(u) = f(x), \quad a < x < b,$$
$$B_1(u) = 0, \quad B_2(u) = 0$$

has the solution

$$u(x) = \int_a^b G(x, y)\, f(y)\, dy$$

where $B_1(u) = 0$ and $B_2(u) = 0$ are the boundary conditions.

Prove that

$$v(x) = A\,u_1(x) + B\,u_2(x) + u(x)$$

with

$$A = \frac{B_2(u_2)\alpha - B_1(u_2)\beta}{\Omega(u_1, u_2)}, \quad B = \frac{-B_2(u_1)\alpha + B_1(u_1)\beta}{\Omega(u_1, u_2)}$$

and $\Omega(u_1, u_2) = B_1(u_1)\,B_2(u_2) - B_1(u_2)\,B_2(u_1)$, gives the solution of the inhomogeneous boundary value problem with inhomogeneous boundary values

$$L(v) = f(x), \quad a < x < b,$$
$$B_1(v) = \alpha, \quad B_2(v) = \beta$$

[25]**7**. Consider the boundary value problems

$$(ii)\ \ \varepsilon y'' + y' + y^2 = 0,\ t > 0, y(0) = 0,\ \varepsilon\,y'(0) = 1.$$

Use the singular perturbation methods to obtain a uniformly valid approximate solution.

[25]**8**. Find the first two dominating terms of the following integral

$$\int_0^\infty \frac{\sin^2 t}{t^2}\, e^{-\lambda t}\, dt$$

where $\lambda >> 1$.

Appendix:
1) $\Gamma(z+1) = \int_0^\infty x^z\, e^{-x}\, dx$
2) $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$

SOLUTIONS

a) $(n+1)\, P_{n+1}(x) = \alpha\, x P_n(x) - n P_{n-1}(x)$

comparing the coefficient of x^{n+1} in both sides we get

$$(n+1)\, k_{n+1} = \alpha\, k_n.$$

since $k_n = \dfrac{2^n\, \Gamma(n+1/2)}{n!\, \Gamma(1/2)}$, then

$$\cancel{(n+1)} \frac{2^{n+1}\, \Gamma(n+3/2)}{\cancel{(n+1)!}\, \cancel{\Gamma(1/2)}} = \alpha\, \frac{2^n\, \Gamma(n+1/2)}{\cancel{n!}\, \cancel{\Gamma(1/2)}}$$

$$2\, \Gamma(n+1/2+1) = \alpha\, \Gamma(n+1/2)$$

$$2\,(n+1/2) = \alpha \quad \Rightarrow \quad \underline{\underline{\alpha = 2n+1.}}$$

b) We know that

$$\int_{-1}^{1} P_0(x)\, P_n(x)\, dx = 0 \qquad n \geq 1.$$

or $\displaystyle\int_{-1}^{1} P_n(x)\, dx = 0$

Then, it is clear that $P_n(x)$ must change sign in $[-1, 1]$. This means that

$P_n(x)$ has roots in $[-1,1]$. The number of roots in $[-1,1]$ is n.

<u>proof:</u> let $\quad P_n(x) = \prod_{i=1}^{r} (x-x_i)\, h(x)$

where $h(x)$ does not change sign in $[-1,1]$. We assume that there exist r number of roots $x_1, x_2, \dots, x_r$. On the other hand

$$\int_{-1}^{1} \prod_{i=1}^{r} (x-x_i)\, P_n(x)\, dx = 0, \qquad r<n.$$

which implies

$$\int_{-1}^{1} \left(\prod_{i=1}^{r} (x-x_i)\right)^2 h(x)\, dx = 0$$

but the integrand is either positive definite or negative definite. Hence the assumption $r<n$ is inconsistent $\Rightarrow$ the number of roots must be n.

(2) See the lecture notes (Lecture Notes 7 and 9, page 9.9 example 2)

(3) See the lecture notes (Lecture Notes 11. pages 13 and 19)

(4) $u'' + 6u' + 9u = x, \quad x > 0$

$u(0) = 0, \quad u'(0) = 1.$

Solution: let $u(x) = e^{-3x} y(x) \Rightarrow$

$$y'' = x e^{3x},$$

$$y(0) = 0, \quad y'(0) = 1.$$

Let $f(x) = x e^{3x} \Rightarrow u_1(x) = 1, \quad u_2(x) = x$

$$y(x) = x + \int_0^\infty G(x,y) f(y)\, dy.$$

$$G(x,y) = \begin{cases} a_1 x + b_1, & x \le y \\ a_2 x + b_2, & x \geq y \end{cases}$$

$$\left.\begin{array}{ll} \text{BCs} \Rightarrow & a_1 = b_1 = 0 \\ \text{Continuity} \Rightarrow & b_2 = -a_2 y \\ \text{Jump cond.} \Rightarrow & a_2 = 1 \end{array}\right\} \Rightarrow G(x,y) = \begin{cases} 0, & x \le y \\ x - y, & x \geq y \end{cases}$$

$$\Rightarrow \quad y(x) = x + \int_0^x G(x,y) f(y)\, dy + \cancel{\int_x^\infty G(x,y) f(y)\, dy}^{\,0}$$

$$= x + \int_0^x (x-y) y e^{3y}\, dy = x + \frac{x}{9} e^{3x} - \frac{2}{27} e^{3x} + \frac{2}{27} + x/9$$

$$\Rightarrow \quad u(x) = x e^{-3x} + \frac{2}{27} e^{-3x} + \frac{x}{9} - \frac{2}{27} + \frac{x}{9} e^{-3x}$$

$$= \frac{10}{9} x e^{-3x} + \frac{2}{27} e^{-3x} + \frac{x}{9} - 2/27$$

5 $L = \dot{z}^2 + a^2 \dot{\phi}^2$,

$$E_z(L) = \frac{\partial L}{\partial z} - \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{z}}\right) = 0 \Rightarrow \ddot{z} = 0$$

$z = \alpha_1 s + \alpha_2$, α_1 and α_2 are constants.

$$E_\phi(L) = \frac{\partial L}{\partial \phi} - \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0 \Rightarrow \ddot{\phi} = 0$$

$$\phi = \beta_1 s + \beta_2$$

i) if $\alpha_1 = 0$ then $z = \alpha_2$ is a circle on cylinder

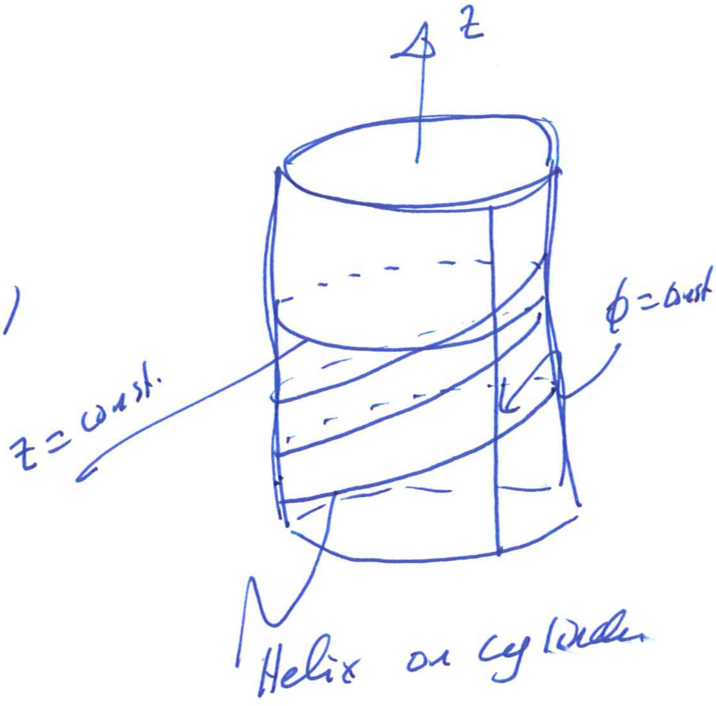
ii) if $\beta_1 = 0$ then $\phi = \beta_2$ is a straight line parallel to z-axis

iii) if $\alpha_1 \neq 0$, $\beta_1 \neq 0 \Rightarrow$

$z = \alpha\phi + \beta$ (α, β are constants)

gives helix on cylinder

Hence the curve connecting A and B is any one of these curves



6 See lecture notes (Lecture Notes 7 and 8)

7 a) Inner-zone : $t = \delta \tau$

$\Rightarrow \frac{\varepsilon}{\delta^2} y_{\tau\tau} + \frac{1}{\delta} y_\tau + y^2 = 0 \Rightarrow \delta = \varepsilon$ gives the

only possible case $\Rightarrow$

$$y_{\tau\tau} + y_\tau + \varepsilon y^2 = 0$$

$$y(\tau) = A + B e^{-\tau} \quad \text{or} \quad y(t) = A + B e^{-t/\varepsilon}$$

$$y(0) = 0 \Rightarrow B = -A \Rightarrow y(t) = A(1 - e^{-t/\varepsilon})$$

$$\varepsilon y'(0) = 1 \Rightarrow y'(0) = A/\varepsilon \Rightarrow A = 1.$$

$$y_{in}(t) = 1 - e^{-t/\varepsilon}$$

b) y_{out}: $y' + y^2 = 0$, $y = \frac{1}{t - t_0}$

c) Matching: let $t = \sqrt{\varepsilon}\, \eta$

$$\lim_{\varepsilon \to 0} y_{in} = 1, \quad \lim_{\varepsilon \to 0} y_{out} = -\frac{1}{t_0} \Rightarrow t_0 = -1$$

$$\Rightarrow y_{out} = \frac{1}{t+1}$$

c) $y_{app} = y_{in} + y_{out} - 1$

$$= \frac{1}{t+1} - e^{-t/\varepsilon}$$

d) residue function $r(t,\varepsilon) = \varepsilon y_{ap}'' + y_{ap}' + y_{ap}^2$

$$\Rightarrow r(t,\varepsilon) = \varepsilon\left(\frac{2}{(t+1)^3} - \frac{1}{\varepsilon^2} e^{-t/\varepsilon}\right) - \frac{1}{(t+1)^2} + \frac{1}{\varepsilon} e^{-t/\varepsilon}$$

$$+ \left(\frac{1}{t+1} - e^{-t/\varepsilon}\right)^2$$

$$= \frac{2\varepsilon}{(t+1)^3} - \frac{2}{t+1} e^{-t/\varepsilon} + e^{-2t/\varepsilon}$$

if $t > T > 0$ then $\lim_{\varepsilon \to 0} r(t,\varepsilon) = 0$

but for $t \to 0$ $\lim_{\varepsilon \to 0} r(t,\varepsilon) \neq 0$

$y_{ap}(0) = 0$, $\varepsilon y_{ap}'(0) = -\varepsilon + e^{-t/\varepsilon} \to 0$

y_{ap} is valid for $t \geq T > 0$

but not uniformly valid for $t > 0$

8

$$\int_0^{\infty} \frac{\sin^2 t}{t^2} e^{-\lambda t} dt$$

$$= \int_0^{T} \frac{\sin^2 t}{t^2} e^{-\lambda t} dt + \int_T^{\infty} \frac{\sin^2 t}{t^2} e^{-\lambda t}$$

i) $\int_T^{\infty} \frac{\sin^2 t}{t^2} e^{-\lambda t} dt < \int_T^{\infty} e^{-\lambda t} dt = \frac{1}{\lambda} e^{-\lambda T}$

= EST (exponentially small term)

ii)

$$\Rightarrow \int_0^{\infty} \frac{\sin^2 t}{t^2} e^{-\lambda t} dt = \int_0^{T} \frac{\sin^2 t}{t^2} e^{-\lambda t} dt + EST$$

$$= \int_0^{T} \frac{1}{t^2} \left(t - \frac{1}{3!} t^3 + \dots\right)^2 e^{-\lambda t} dt + EST$$

$$= \int_0^{T} \frac{1}{t^2} \left(t^2 - \frac{1}{3} t^4 + \dots\right) e^{-\lambda t} dt + EST$$

$$= \int_0^{T} \left(1 - \frac{1}{3} t^2\right) e^{-\lambda t} dt + EST$$

$$= -\frac{1}{\lambda} e^{-\lambda t} \Big|_0^{T} - \frac{1}{3} \int_0^{T} t^2 e^{-\lambda t} dt + EST$$

$$= \frac{1}{\lambda} - \frac{2}{3\lambda^3} + EST$$