MATH 543

METHODS OF APPLIED MATHEMATICS I
First Midterm Exam : SOLUTIONS

March 20, 2017
Monday 10.40-12.30

PROBLEMS: Choose any three of the following four questions:

1[35]. **(a)** Assume that there exists an orthonormal basis $|e_i>$, $(i = 1, 2, \cdots)$ in $L^2_w(a, b)$. Then , for any $|f> \in L^2_w(a, b)$, the sequence of vectors

$$|f_k> = \sum_{i=1}^{k} f^i \, |e_i>$$

with

$$f^i = < e_i|f >, \quad i = 1, 2, \cdots$$

has $|f>$ as the limit vector in the sense that

$$\lim_{k\to\infty} \rho(|f>, |f_k>) = 0.$$

.
(b) State and prove Parseval's Theorem.

SOLUTION : See Lecture Notes 1-2 page 35

2[35]. Consider the following sequence:

$$h_n(x) = \begin{cases} 0 & \text{if } x \le \frac{-1}{n} \\ \frac{(nx+1)}{2} & \text{if } \frac{-1}{n} \le x \le \frac{1}{n} \\ 1 & \text{if } x \ge \frac{1}{n} \end{cases}$$

(a) Prove that the sequence $h_n(x)$ defines the step function $\theta(x)$ and
(b) $\frac{dh_n(x)}{dx}$ defines the Dirac delta function $\delta(x)$. Hence formally we may write that $\frac{d\theta(x)}{dx} = \delta(x)$.

SOLUTION: See Lecture Notes 6 page 91

[35]**3**. Show that every second order linear differential operator $L = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x)$ (where $a(x) > 0$) with real coefficients is self-adjoint with respect to the weight function $w(x) = \frac{1}{a(x)}\, e^{\int^x \frac{b(\xi)}{a(\xi)}\, d\xi}$.

SOLUTION: See Lecture Notes 7 page 5-6

[35]**4**. $C_{(n)}(x)$'s are the classical orthogonal polynomials of degree n with

$$C_{(n)}(x) = k_n\, x^n + k'_n\, x^{n-1} + p_{(n-2)}(x), \tag{1}$$

$$\int_a^b w(x)(C_{(n)}(x))^2 dx = h_n \tag{2}$$

$$C_{(n+1)}(x) - \frac{k_{n+1}}{k_n}\, x\, C_{(n)}(x) = \left(\frac{k'_{n+1}}{k_n} - \frac{k_{n+1}}{k_n^2} k'_n\right) C_{(n)}(x)$$
$$-\frac{k_{n-1}k_{n+1}}{k_n^2}\, \frac{h_n}{h_{n-1}}\, C_{(n-1)}(x) \tag{3}$$

where $p_{(n-2)}(x)$ is a polynomial of degree $n-2$. Find the following integrals

$$(a)\ \int_a^b w(x) x C_{(n-1)}(x) C_{(n)} dx,$$
$$(b)\ \int_a^b w(x) C_{(n)}(x)\, C_{(m)}(x) C_{(n+m)}(x) dx,$$
$$(c)\ \int_a^b w(x) x\, (C_{(n)})^2\, dx$$

in terms of k_n, k'_n and h_n.

SOLUTION: (a). Multiplying both sides of (3) by $wC_{(n-1)}$ and integrating from a to b we get

$$-\frac{k_{n+1}}{k_n} \int_a^b w(x) x C_{(n-1)}(x) C_{(n)} dx = -\frac{k_{n-1}k_{n+1}}{k_n^2}\, h_n \tag{4}$$

Hence

$$\int_a^b w(x) x C_{(n-1)}(x) C_{(n)} dx = -\frac{k_{n-1}}{k_n}\, h_n \tag{5}$$

(b).

$$\int_a^b w(x)C_{(n)}(x)\,C_{(m)}(x)C_{(n+m)}(x)dx = \int_a^b w(x)[k_n\,k_m x^{n+m} + p_{n+m-1}(x)]C_{(n+m)}(x)dx =$$
$$\int_a^b w(x)[k_n\,k_m x^{n+m}]C_{(n+m)}(x)dx = \frac{k_n\,k_m}{k_{n+m}}\,h_{n+m} \tag{6}$$

(c). Multiplying both sides of (3) by $wC_{(n)}$ and integrating from a to b we get

$$-\frac{k_{n+1}}{k_n}\int_a^b x\,(C_{(n)}(x)(^2\,dx = \left(\frac{k'_{n+1}}{k_n} - \frac{k_{n+1}}{k_n^2}k'_n\right)h_n \tag{7}$$

Hence

$$\int_a^b x\,(C_{(n)}(x))^2\,dx = \left(-\frac{k'_{n+1}}{k_n+1} + \frac{1}{k_n}k'_n\right)h_n \tag{8}$$