MATH 543
METHODS OF APPLIED MATHEMATICS I
Final Exam

January 4, 2016
Monday 15:30-17:30

PROBLEMS: Choose any three of the following four questions:

1[35].Let $y'' + \varepsilon(y')^2 + y = 0$ for $t > 0$ and $\varepsilon << 1$ with $y(0) = 1,\;\; y'(0) = 0$. Find a first order approximation of this problem and discuss the validity of the approximation.

2[35].**(2.a)** Let

$$I(\lambda) = \int_a^b f(t)\, e^{\lambda g(t)}\, dt$$

where f is continuous and the maximum of g occurs at $t = b$ one of the end point of the interval with $g'(t) > 0$ for all $t \in [a, b]$. Then show that

$$I(\lambda) \sim \frac{f(b)\, e^{\lambda g(b)}}{\lambda g'(b)}$$

for $\lambda >> 1$. **(2.b)** Find the dominant term of

$$J(\lambda) = \int_0^{\pi/4} \cos t\, e^{\lambda \sin^2 t}\, dt$$

for $\lambda >> 1$.

3[35].Find the WKB approximation of the following eigenvalue problem

$$y'' + \lambda(\cosh^4 x)\; y = 0, \quad y(0) = 0, \quad y(1) = 0,$$

for large λ.

4[35]. Consider the following boundary value problem

$$\begin{aligned} \varepsilon y'' + y' + y &= 0, \;\; (t > 0, \varepsilon << 1) \\ y(0) &= 0, \;\; \varepsilon\, y'(0) = 1. \end{aligned}$$

Use singular perturbation methods to obtain a uniform approximate solution of this boundary value problem.

Solutions of the Final 2015 problems

1. $y'' + y + \varepsilon y'^2 = 0, \quad t > 0, \quad 0 < \varepsilon \ll 1.$

$y(0) = 1, \quad y'(0) = 0.$

Solution by Poincaré - Lindstead Method. Let $\tau = \omega t$

$$\omega^2 y_{\tau\tau} + y + \varepsilon \omega^2 y_\tau^2 = 0$$

First order approximation

$$y(\tau) = y_0(\tau) + \varepsilon y_1(\tau) + \cdots$$
$$\omega = 1 + \varepsilon \omega_2 + \cdots$$

$\Rightarrow$

$$(1 + \varepsilon\omega_1 + \cdots)^2 (y_0'' + \varepsilon y_1'' + \cdots) + (y_0 + \varepsilon y_1 + \cdots)$$
$$+ \varepsilon (1 + \omega_1 \varepsilon + \cdots)^2 (y_0' + \varepsilon y_1' + \cdots)^2 = 0$$

i) $y_0'' + y_0 = 0, \quad y_0(0) = 1, \quad y_0(0) = 0.$

$$y_0(\tau) = \cos\tau.$$

ii) $y_1'' + y_1 = -2\omega_1 \cos\tau - \sin^2\tau, \quad y_1(0) = 0, \; y_1'(0) = 0$

To prevent the secular term at the first order approximation we let $\omega_1 = 0$. Then

$$y_1'' + y_1 = -\frac{1}{2} + \frac{1}{2}\cos 2\tau.$$

$$y_1(\tau) = -\frac{1}{2} - \frac{1}{6}\cos 2\tau + \frac{2}{3}\cos\tau.$$

$$y_{ap} = \cos\tau + \varepsilon\left(-\frac{1}{2} - \frac{1}{6}\cos 2\tau + \frac{2}{3}\cos\tau\right) + \cdots$$

$$\omega = 1$$

The residue function

$$r(\varepsilon, y_{ap}) = y_{ap}'' + y_{ap} + \varepsilon\, y_{ap}'^{\,2}$$

$$= -\cos\tau + \varepsilon\left(+\frac{2}{3}\cos 2\tau - \frac{2}{3}\cos\tau\right)$$

$$+ \cos\tau + \varepsilon\left(-\frac{1}{2} - \frac{1}{6}\cos 2\tau + \frac{2}{3}\cos\tau\right)$$

$$+ \varepsilon\sin^2\tau \quad + O(\varepsilon^2).$$

$$= O(\varepsilon^2).$$

$$\Rightarrow \lim_{\varepsilon \to 0} r(\varepsilon, y_{ap}) = 0.$$

2.a) $$I = \int_a^b f(t)\, e^{\lambda g(t)}\, dt$$

$$g(t) = g(b) + g'(b)(t-b) + \frac{1}{2} g''(b)(t-b)^2 + \cdots$$

$$\Rightarrow I = e^{\lambda g(b)} \int_a^b f(t)\, e^{\lambda g'(b)(t-b) + \frac{\lambda}{2} g''(b)(t-b)^2 + \cdots}\, dt$$

let $\lambda g'(b)(t-b) = u$, $g'(b) > 0$

$$dt = \frac{du}{\lambda g'(b)}$$

$$\Rightarrow I = \frac{e^{\lambda g(b)}}{\lambda g'(b)} \int_{\lambda g'(b)(a-b)}^{0} f\left(b + \frac{u}{\lambda g'(b)}\right) e^{u + O(\frac{1}{\lambda})}\, du$$

as $\lambda \to \infty$ and since f is continuous

then

$$I \sim \frac{e^{\lambda g(b)}}{\lambda g'(b)} f(b) \int_{-\lambda g'(b-a)}^{0} e^{u}\, du$$

$$\sim \frac{e^{\lambda g(b)} f(b)}{\lambda g'(b)} \left[1 - e^{-\lambda g'(b-a)}\right]$$

$$\sim \frac{e^{\lambda g(b)} f(b)}{\lambda g'(b)}$$

2b) $I = \int_0^{\pi/4} \cos t \, e^{\lambda \sin^2 t} \, dt \quad , \quad \lambda \gg 1$

$f(t) = \cos t \Rightarrow f(\pi/4) = \cos \pi/4 = 1/\sqrt{2}$

$g(t) = \sin^2 t \; , \; t \in [0, \pi/4]$, since g has its maximum at $t = \pi/4$. we can use (2.a).

$g'(t) = 2 \sin t \cos t = \sin 2t \; , \; g'(\pi/4) = \sin \pi/2 = 1$

$$\Rightarrow \int_0^{\pi/4} \cos t \, e^{\lambda \sin^2 t} \, dt \approx \frac{1/\sqrt{2} \, e^{\lambda/2}}{\lambda} = \frac{e^{\lambda/2}}{\sqrt{2} \, \lambda}$$

3) $y'' + \lambda \cosh^4 x \, y = 0$, $\lambda \gg 1$

$y(0) = 0$, $y(1) = 0$

let $y = e^{\phi/\varepsilon}$ and $\lambda = \frac{1}{\varepsilon^2}$

$\Rightarrow \varepsilon \phi'' + \phi'^2 + \cosh^4 x = 0$

Let $\phi' = u \Rightarrow \varepsilon u' + u^2 + \cosh^4 x = 0$

$u = u_0 + \varepsilon u_1 + \cdots$

$\varepsilon (u_0' + \varepsilon u_1' + \cdots) + (u_0 + \varepsilon u_1 + \cdots)^2 + \cosh^4 x = 0$

$u_0^2 + \cosh^4 x = 0 \Rightarrow u_0 = \pm i \cosh^2 x$

$u_0' + 2 u_0 u_1 = 0 \Rightarrow u_1 = -\frac{1}{2} \frac{u_0'}{u_0} = -\tanh x$

$\phi' = \pm i \cosh^2 x - \varepsilon \tanh x$

$\phi = \pm i \int^x \frac{1}{4}(e^{2x} + e^{-2x} + 2)\, dx - \varepsilon \ln(\cosh x) + \text{const}$

$= \pm i \frac{1}{8}(e^{2x} - e^{-2x} + 4x) - \varepsilon \ln(\cosh x) + \cdots$

$\Rightarrow y(x) = \frac{1}{\cosh x} \left[A \cos\left[\frac{\sqrt{\lambda}}{8}(e^{2x} - e^{-2x} + 4x)\right] + B \sin\left[\frac{\sqrt{\lambda}}{8}(e^{2x} - e^{-2x} + 4x)\right]\right]$

$y(0) = 0 \qquad A = 0$

$y(1) = 0 \qquad \frac{\sqrt{\lambda}}{8}\left(e^{2} - e^{-2} + 4\right) = n\pi \;, \quad n = 1, 2, \ldots$

$$\lambda_n = \frac{64\, n^2 \pi^2}{\left(e^{2} - e^{-2} + 4\right)^2}$$

$$y_n(x) = \frac{B}{\cosh x} \sin\left[\frac{\sqrt{\lambda_n}}{8}\left(e^{2x} - e^{-2x} + 4\right)\right]$$

$n = 1, 2, \ldots$

B is a constant.

4) $\varepsilon y'' + y' + y = 0, \quad t > 0, \quad 0 < \varepsilon \ll 1$

$y(0) = 0, \ \varepsilon y'(0) = 1$

This is a singular perturbation problem.

i) Inner zone: $t = \delta \tau$.

$$\frac{\varepsilon}{\delta^2} y_{\tau\tau} + \frac{1}{\delta} y_\tau + y = 0$$

Comparison:

i) first term with the second $\Rightarrow \delta = \varepsilon$

ii) first term with the third term $\delta = \sqrt{\varepsilon}$

but the second term is large. Not suitable

$\delta = \varepsilon$

$$y_{\tau\tau} + y_\tau + \varepsilon y = 0$$

$$y(0) = 0, \quad y_\tau(0) = 1$$

$$y = y_0 + \varepsilon y_1 + \dots$$

$$(y_0'' + \varepsilon y_1'' + \dots) + (y_0' + \varepsilon y_1' + \dots) + \varepsilon(y_0 + \varepsilon y_1 + \dots) = 0$$

a) $y_0'' + y_0' = 0 \qquad y_0(0) = 0, \ y_0'(0) = 1$

$y_0 = A + B e^{-\tau} \qquad \Downarrow$

$A + B = 0 \qquad -B = 1$

$A = 1$

$$y_0(\tau) = 1 - e^{-\tau}$$

$$= 1 - e^{-t/\varepsilon}$$

ii) Outer zone $y_0' + y_0 = 0$

$$y_0 = c\,e^{-t} \quad , \quad c \text{ is arbitrary}$$

iii) Matching zone $t = \sqrt{\varepsilon}\, y$.

$$\lim_{\varepsilon \to 0} y_{in} = \lim_{\varepsilon \to} y_{out}$$

$$c = 1$$

iv) $y_{cp} = y_{in} + y_{out} - y_{matching\ zone}$

$$= 1 - e^{-t/\varepsilon} + e^{-t} - 1$$

$$= e^{-t} - e^{-t/\varepsilon}$$

v) Uniformly valid approximation

$$r(\varepsilon, t) = \varepsilon y_{cp}'' + y_{cp}' + y_{cp} = \varepsilon e^{-t} - \cancel{\frac{1}{\varepsilon} e^{-t/\varepsilon}}$$

$$- \cancel{e^{-t}} + \cancel{\frac{1}{\varepsilon} e^{-t/\varepsilon}} + \cancel{e^{-t}} - e^{-t/\varepsilon}$$

$$= \varepsilon e^{-t} - e^{-t/\varepsilon}$$

$$\lim_{\varepsilon \to \infty} r(\varepsilon, t) = 0 \quad ✓ \quad , \quad y_{cp}(0) = 0 \quad ✓$$

$$\varepsilon y_{cp}'(0) = -\varepsilon + 1 \quad ✓$$