MATH 543

METHODS OF APPLIED MATHEMATICS I
Second Midterm Exam SOLUTIONS

December 12, 2014
Friday 08.40-10.30

PROBLEMS:Choose any three of the following four questions:

1[35]. Solve the following boundary value problem by the use of Green's function metod

$$(xu')' - \frac{n^2}{x}\,u = k\,x^{m-1}, \quad x \in (0,1),$$
$$\lim_{x\to 0} u(x) = 0, \quad u(1) = 0,$$

where n and m are positive integers and k is any real number. What happens when $m = n$?.

Solution: Green's Function function of the problem is

$$G(x,y) = \begin{cases} \frac{1}{2n}[(xy)^n - (x/y)^n] & x \le y \\ \frac{1}{2n}[(xy)^n - (y/x)^n] & x \ge y \end{cases}$$

and the solution is

$$u(x) = \frac{k}{m^2 - n^2}(x^m - x^n) \tag{1}$$

If $m = n$, then

$$u(x) = \frac{k}{2n}\,x^n\,\ln x \tag{2}$$

2[35]. Let $z = z_0$ be a regular singular point of the differential equation $u'' + p(z)\,u' + q(z)\,u = 0$ with indices r_1 and r_2. Show that the solution of this differential equation about $z = z_0$ corresponding to the index r_2 has the form

$$u(z) = A\,u_1(z)\,\ln(z - z_0) + (z - z_0)^{r_2} \sum_{n=0}^{\infty} C_n\,(z - z_0)^n$$

when $r_1 - r_2$ is a positive integer or zero. Here A and C_n's are constants. **HINT:** If u_1 is given then $u_2 = Cu_1 \int^x \frac{dx'}{u_1^2(x')} e^{-\int^{x'} \frac{b(\xi)}{a(x^\xi)}\, d\xi}$ where C is a constant.

Solution: Se Lecture Notes

3[35]. The hypergeometric differential equation is given as

$$z(1-z)\,u'' + [c - (1+a+b)\,z]\,u' - ab\,u = 0,$$

where a, b, and c are constants. **(a)**: Find the regular singular points and their corresponding indices of this equation. **(b)**: Find the solution of the above equation about $z = 0$ when $b - c$ is not an integer or zero.

Solution: Se Lecture Notes

4[35]. Prove that every linear second order differential operator $L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c$ with real coefficients a, b, and c is self adjoint provided the weight function $w(x)$ is chosen properly

$$w(x) = \frac{p}{a}, \qquad \frac{p'}{p} = \frac{b(x)}{a(x)}, \tag{3}$$

where the differential operator takes the form

$$L_x u = \frac{1}{w}(\frac{d}{dx}\, p(x)\frac{du}{dx}) + cu, \tag{4}$$

and the surface term Q is given by

$$Q(u, \bar{v}) = p(x)\,(\bar{v}\frac{du}{dx} - u\frac{d\bar{v}}{dx}), \tag{5}$$

Solution: Se Lecture Notes