MATH 543

METHODS OF APPLIED MATHEMATICS I
First Midterm Exam

October 31, 2014
Friday 08.40-10.30

PROBLEMS:Choose any three of the following four questions:

1[35]. **(a)** Assume that there exists an orthonormal basis $|e_i>$, $(i = 1, 2, \cdots)$ in $L^2_w(a, b)$. Then , for any $|f> \in L^2_w(a, b)$, the sequence of vectors

$$|f_k> = \sum_{i=1}^{k} f^i \, |e_i>$$

with

$$f^i = < e_i|f >, \quad i = 1, 2, \cdots$$

has $|f>$ as the limit vector in the sense that

$$\lim_{k\to\infty} \rho(|f>, |f_k>) = 0.$$

(b) Prove that any orthogonal family $|e_i >, \quad i = 1, 2, = \cdots$ in $L^2_w(a, b)$ is linearly independent. **(c)** If $(e_1, e_2, \cdots, e_n)$ is a finite orthonormal family of functions in $L^2_w(a, b)$ and $f \in L^2_w(a, b)$, then

$$||f - \sum_{k=1}^{n} f^k \, |e_k > ||$$

has its minimum value when $f^i = < f|e_i >, \quad i = 1, 2, \cdots$ and

$$\sum_{k=1}^{n} | < f|e_k > |^2 \leq ||f||^2$$

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2[35]. **(a)** Let $e_n(x)$, $(n = 0, 1, 2, \cdots)$ be one of the classical orthonormal polynomials, i.e., $e_n(x) = \frac{C_n(x)}{||C_n(x)||}$, with weight function $w(x)$ and $x \in [a, b]$. Prove that the sequence

$$h_n(x) = w(x) \sum_{k=0}^{n} e_k(x)\, e_k(0)$$

where $0 \in [a, b]$ defines the delta function $\delta(x)$. [**Hint**: You have to prove (a) $\int_a^b \delta(x)\, dx = 1$, and (b) $\int_a^b \delta(x) g(x)\, dx = g(0)$ for any test function $g(x)$, and take $e_0(x) = 1$]
(b) Consider the following sequence:

$$h_n(x) = \begin{cases} 0 & \text{if } x \le \frac{-1}{n} \\ \frac{(nx+1)}{2} & \text{if } \frac{-1}{n} \le x \le \frac{1}{n} \\ 1 & \text{if } x \ge \frac{1}{n} \end{cases}$$

Prove that the sequence $\frac{dh_n(x)}{dx}$ defines the Dirac delta function $\delta(x)$. Hence formally we may write that $\frac{d\theta(x)}{dx} = \delta(x)$.

[35]**3**. Let

$$C_n(x) = \frac{1}{K_n\, w(x)} \frac{d^n}{dx^n} (w(x) s(x)^n), \quad n = 0, 1, 2, \cdots, x \in [a, b]$$

where
(a) $w(x) > 0$ for all $x \in [a, b]$.
(b) $s(x)$ is a polynomial of degree ≤ 2 with real distinct roots.
(c) $C_1(x)$ is a polynomial of degree 1.
(d) $w(a)s(a) = w(b)s(b) = 0$ (boundary conditions),
(e) K_n's are constants.

Classical orthogonal polynomials C_n's satisfy
e) recursion relation

$$C_{n+1}(x) - \frac{k_{n+1}}{k_n} xC_n = \alpha_n\, C_n + \beta_n\, C_{n-1}$$

where

$$\alpha_n = \frac{k'_{n+1}}{k_n} - \frac{k_{n+1}}{k_n^2} k'_n, \quad \beta_n = -\frac{k_{n+1}\, k_{n-1}}{k_n^2} \frac{h_n}{h_{n-1}}$$

and
f) a second order ODE

$$\frac{1}{w}\frac{d}{dx}\left(ws\frac{dC_n}{dx}\right) = \lambda\, C_n$$

3.1. Show that

$$\lambda = n[K_1\,\frac{dC_1}{dx} + \frac{1}{2}(n-1)\,\frac{d^2s}{dx^2}]$$

3.2. Find $< xC_n, C_m >$ in terms of k, k' and h.

[35]**4**. Give formal definitions of the following in a (linear) function space
a) Inner product, norm and metric interms of the inner product
b) Completeness
c) Bessel's inequality
d) Basis of a function space
e) Parseval's realtion
f) Weiertrass theorem
g) Generalized functions