MATH 543
METHODS OF APPLIED MATHEMATICS I
FINAL EXAM

January 11, 2012
Wednesday 15:30-17:30, SAZ-20

QUESTIONS: Solve 4 problems.

1 [25]. Find an approximate solution of the differential equation

$$\epsilon \frac{d^2u}{dt^2} + \frac{du}{dt} + \epsilon u^2 = 0, \quad t > 0 \tag{1}$$

$(0 < \epsilon << 1)$ with the initial conditions $u(0) = 0$ and $\epsilon\, u'(0) = 1$. Discuss whether your solution is a uniformly valid approximation.

2 [25]. The hypergeometric equation is

$$z(1-z)\frac{d^2u}{dt^2} + [c - z(a+b+1)]\,\frac{du}{dt} - ab\,u = 0 \tag{2}$$

where a, b and c are constants. The solution of this equation about its regular singular point $z = 0$ with index $r = 0$ is $u(z) = F(a, b, c; z)$, the hypergeometric function. Find the general solution of this equation about its regular singular point $z = \infty$.

3 [25]. Let

$$z\frac{d^2u}{dz^2} + (1-z)\frac{du}{dz} - u = 0. \tag{3}$$

Using the Frobenius method find the solution of this equation about all its regular singular points.

4 [25]. (a) Let $u_n(x)$, $(n = 0, 1, 2, \cdots)$ be one of the classical orthonormal polynomials with weight function $w(x)$ and $x \in [a, b]$. Prove that the sequence

$$h_n(x) = w(x) \sum_{k=0}^{n} u_k(x)\, u_k(x_0)$$

where $x_0 \in [a, b]$ defines the delta function $\delta(x - x_0)$.

5 **[25]**. Let

$$I(\lambda) = \int_a^b f(t) e^{\lambda g(t)}\, dt$$

where f is continuous and g is sufficiently smooth and has a unique maximum at a point $x = c$ in (a, b). Then

$$I(\lambda) \sim\ f(c) e^{\lambda g(c)} \sqrt{\frac{-2\pi}{\lambda g''(c)}}, \quad \lambda >> 1$$

If the point $x = c$ happens to be at one of the end points a or b then show that the right hand side of the above expression should be divided by 2.

6 **[25]**. Let z_0 be an ordinary point of the differential equation

$$\frac{d^2u}{dz^2} + p(z)\, \frac{du}{dz} + q(z)\, u = 0. \tag{4}$$

with the boundary conditions $u(z_0) = a$ and $\frac{du}{dz}(z_0) = b$, where $p(z)$ and $q(z)$ are analytic functions in a region R, except perhaps at an enumerable number of points of R where these functions may have isolated singularities. Here a and b are given numbers. Show that this boundary value problem has unique solutions in the neighborhood of z_0.