MATH 543
METHODS OF APPLIED MATHEMATICS I
Second Midterm Exam

December 13, 2011
Tuesday 13:40-15:30, SAZ-02

QUESTIONS: Solve at least 4 problems.

1 [30]. Let $L_x = a(x)\,\frac{d^2}{dx^2} + b(x)\,\frac{d}{dx} + c(x)$ where a, b and c are real continuous functions for all $x \in (a,b)$. Prove that every linear formal differential operator,L_x, of second order with real coefficients is self-adjoint, provided that the weight function $w(x)$ is chosen as

$$w(x) = \frac{1}{a(x)}\, e^{\int_{x_0}^{x} dx' \frac{b(x')}{a(x')}} \tag{1}$$

and $a(x) > 0$.
2 [30]. Let a boundary value problem is given by

$$L_x(u) = f(x), \quad x \in (a,b), \tag{2}$$
$$B_1(u) = B_2(u) = 0 \tag{3}$$

where L_x is a second order, linear, real and a hermitian operator, B_1 and B_2 are linear operators defining the boundary conditions. Show that

$$u(x) = \int_a^b w(y)\, f(y) G(x,y) dy \tag{4}$$

where $G(x,y)$ is the Green's function of the problem.
3 [30]. Prove that the Green's function of the above boundary value problem exists and unique provided that there exists no nontrivial solution of the homogenous equation satisfying both boundary conditions.

4 [30]. Solve the following boundary value problem by using the the method of Green's function.

$$\frac{d^2u}{dx^2} - 4\,u = e^x, \quad x \in (0, a), \tag{5}$$

$$u(0) = \lambda_1, \quad u(a) = \lambda_2 \tag{6}$$

where λ_1 and λ_2 are given real numbers.

5 [30]. Let L_x is a second order, linear, real and a hermitian operator. Show that the eigenvalue problem

$$L_x(u_n) = -\lambda_n\, u_n, \quad x \in (a, b), \tag{7}$$

$$u_n(a) = u_n(b) = 0 \tag{8}$$

For all $n = 0, 1, 2\cdots$, **(5.1)** has real eigenvalues and **(5.2)** the eigenfunction for different eigenvalues are orthogonal with respect to the L^2-inner product. **(5.3)** If $L_x = \frac{1}{w(x)}\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + c(x)$ with $p(x) > 0$. Then show that the eigenvalues are non-negative if $c(x) \leq 0$ for all $x \in (a, b)$.