MATH 543

METHODS OF APPLIED MATHEMATICS I
Final Exam: SOLUTIONS

January 6, 2014
Monday 09.00-11.00 (SBZ-11)

PROBLEMS:Choose any three of the following problems

1[40]. *Regular Perurbation Method*: Using a perturbation method find a uniformly valid first order approximate solution of the following initial value problem.

$$y'' + 9y = 3\,\varepsilon y^3,\ t > 0,\quad \varepsilon << 1$$
$$y(0) = 0,\quad y'(0) = 1$$

Solution: Use the Poincare-Lindsteadt Method. let $t = \frac{\tau}{w(\varepsilon)}$ then

$$w^2\, y_{\tau\tau} + 9y = 3\,\varepsilon y^3,\ \tau > 0,\quad \varepsilon << 1,$$
$$y(0) = 0,\quad w\, y_\tau(0) = 1$$

Now let $w = 3 + \varepsilon\, w_1 + \cdots$ and $y(\tau, \varepsilon) = y_0(\tau) + \varepsilon\, y_1(\tau)$, then we obtain system of equations

$$y_{0,\tau\tau} + y_0 = 0,\quad y_0(0) = 0,\ 3\, y_{0,\tau}(0) = 1,$$
$$9\, y_{1,\tau\tau} + 9y_1 + 6w_1 y_{0,\tau\tau} = 9y_0^3,\quad y_1(0) = 0,\ w_1\, y_{0,\tau}(0) + 3y_{1,\tau}(0) = 0,$$
$$\cdots, \cdots, \cdots,$$

First one has the solution $y_0 = \frac{1}{3}\sin(\tau)$. To eliminate the secular terms we must take $w_1 = -\frac{3}{8}$ and the second solution becomes

$$y_1(\tau) = -\frac{1}{248}\sin(\tau) + \frac{11}{96}\sin(3\tau)$$

There the two term approximation becomes

$$y_{app}(\tau, \varepsilon) = \frac{1}{3}\sin(\tau) + \varepsilon\left(-\frac{1}{248}\sin(\tau) + \frac{11}{96}\sin(3\tau)\right) + \cdots,$$
$$\tau = (3 - \frac{3}{8}\varepsilon + \cdots)\, t$$

This is a uniformly valid approcimation for all $t > 0$. The residue function $r(\tau, \varepsilon) = w^2\, y_{app,\tau\tau} + 9y_{app} - 3\,\varepsilon y_{app}^3$ is bounded, i.e., $|r(\tau, \varepsilon)| \le \varepsilon^2\, M(\varepsilon)$ for all $t > 0$. Here $M \to M_0$ as $\varepsilon \to 0$. Hence it can be easily shown that $\lim_{\varepsilon \to 0}\, r(\tau, \varepsilon) = 0$.

2[40]. *Solutions in the neighborhood of a regular singular point*: Solve the differential equation $2z^2\, y'' -\, z\, y' + (1 + z)y = 0$ about its regular singular points.

Solution: $z = 0$ is the regular singular point of the DE with $a_0 = -1/2,\ \ b_0 = 1/2$. Hence $r_1 = 1,\ \ r_2 = 1/2$ and

$$a_{n+1} = -\frac{a_n}{(r + n + 1)(2r + 2n - 1) + 1} \quad n \ge 1$$

which leads to

$$\begin{aligned} r_1 &= 1, \quad a_n = \frac{(-1)^n}{[1 \cdot 3 \cdot 5 \cdots (2n + 1)]n!}\, a_0, \quad n \ge 1, \\ r_2 &= 1/2, \quad a_n = \frac{(-1)^n}{[1 \cdot 3 \cdot 5 \cdots (2n - 1)]n!}\, a_0, \quad n \ge 1 \end{aligned}$$

and the corresponding solutions are

$$u_1(z) = z\,[1 + \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{[3 \cdot 5 \cdot 7 \cdots (2n + 1)]n!}\,], \tag{1}$$

$$u_2(z) = \sqrt{z}\,[1 + \sum_{n=1}^{\infty} \frac{(-1)^n\, z^n}{[1 \cdot 3 \cdot 5 \cdots (2n - 1)]n!}\,] \tag{2}$$

The general solution is $u(z) = \alpha u_1(z) + \beta u_2(z)$ where α and β are arbitrary constants.

3[40]. *Fuchsian Differential Equations*: Prove that a second order Fuchsian differential equation having only two regular singular points is equivalent to a differential equation with constant coefficients, hence solvable in terms of the elementary functions sin z,cos z and polynomial in z.

Solution: See the Lecture Notes.

4[40]. *Classical Orthogonal Polynomials*: Find $< xC_n, C_m >$ where $C_n(x)$ is any one of the classical orthogonal polynomial with $x \in [a, b]$. Calculate this inner product for Legendre polynomials.

SOLUTION: In general $C_n(x) = k_n\, x^n + k'_n\, x^{n-1} + \cdots$. We have two distinct cases (i) the case where $n > m$ (without loosing any generality we include the case $m > n$ into this case by symmetry) and (ii) $m = n$.

(i): $n > m$.

$$< xC_n, C_m >= \begin{cases} \Delta, \text{when} \;\; n = m + 1 \\ 0, \text{when} \;\; n > m + 1 \end{cases} \tag{3}$$

where

$$\Delta =< C_n, xC_{n-1} >= k_{n-1} < x^n, C_n >= k_{n-1}\, \frac{h_n}{k_n} \tag{4}$$

For the Legendre polynomials $\Delta = \frac{n}{2(n+1/2)(n-1/2)}$

(ii): $n = m$.

$$< C_{n+1}, C_n >=< k_{n+1}x^{n+1} + k'_{n+1}\, x^n + \cdots, C_n >= 0 \tag{5}$$

Then

$$k_{n+1} \; < x^{n+1}, C_n > + k'_{n+1} \; < x^n, C_n >= 0$$

or

$$< x^{n+1}, C_n >= -\frac{k'_{n+1}}{k_{n+1}} \; < x^n, C_n >$$

This leads to

$$\begin{aligned} < xC_n, C_n > \;\; &= \;\; k_n \; < x^{n+1}, C_n > + k'_n \; < x^n, C_n >, & (6) \\ &= \;\; [-\frac{k'_{n+1}}{k_{n+1}}\, k_n + k'_n] \; < x^n, C_n >, & (7) \\ &= \;\; [-\frac{k'_{n+1}}{k_{n+1}}\, k_n + k'_n]\, \frac{h_n}{k_n} & (8) \end{aligned}$$

For the Legendre polynomials $< xC_n, C_n >= 0$ for all $n = 0, 1, \cdots$

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Appendix:

$$P_n(x) = \frac{2^n\,\Gamma(n+1/2)}{\sqrt{\pi}\,n!}\,x^n + c_n\,x^{n-2} + \cdots\,, \quad h_n = ||P_n||^2 = \frac{1}{n+1/2}$$