MATH 543

METHODS OF APPLIED MATHEMATICS I
Second Midterm Exam: SOLUTIONS

December 12, 2013
Thursday 10.30-12.20

PROBLEMS:

1[40]. *The method of Green's function*: Let $u'' = f(x), \quad x \in (0,1)$ with $u(0) - \alpha\, u'(0) = \gamma_1$ and $u(1) - \beta\, u'(1) = \gamma_2$. Here α, β, γ_1 and γ_2 are given constants and $f(x)$ is a given function. Solve this boundary value problem by the use of Green's function method.

SOLUTION:

$$\begin{aligned} u'' &= f(x) \quad x \in (0,1), \\ B_1(u) &= u(0) - \alpha\, u'(0) = \gamma_1, \\ B_2(u) &= u(1) - \beta\, u'(1) = \gamma_2, \end{aligned}$$

This is a BV problem with inhomogeneous boundary conditions.

(i). We first solve the problem as if it was a BV problem with homogeneous boundary conditions. The the fundamental set is $\{u_1(x) = 1, \quad u_2(x) = x\}$. Then

$$G(x,y) = \begin{cases} a_1 + b_1\, x, & x \le y \\ a_2 + b_2\, x, & x \ge y \end{cases} \tag{1}$$

Using the boundary conditions $B_1(G) = B_2(G) = 0$ and continuity at $x = y$ and the jump conditions we find

$$\begin{aligned} a_1 &= \frac{\alpha\,(\beta - 1 + y)}{\Delta}, \quad a_2 = \frac{(\beta - 1)\,(\alpha + y)}{\Delta}, \\ b_1 &= \frac{\beta - 1 + y}{\Delta}, \quad b_2 = \frac{\alpha + y}{\Delta} \end{aligned}$$

Hence

$$G(x,y) = \begin{cases} \frac{(\beta-1+y)(\alpha+x)}{\Delta} & x \le y \\ \frac{(\alpha+y)(\beta-1+x)}{\Delta} & x \ge y \end{cases} \tag{2}$$

where $\Delta = \alpha - \beta + 1$. Here we assumed that $\Delta \neq 0$. If $\Delta = 0$ a solution $\alpha + x$ of the homegeneous equation satisfies both boundary condtions. In this case the construction of the GF collapses. We shall continue with $\Delta \neq 0$.

(ii). The the solution of the BV value problem with inhomogeneous boundary condtions can be given as

$$u(x) = A_1 + B_1\, x + \int_0^1 G(x,y)\, f(y)\, dy$$

where A_1 and B_1 will be determined from the condtions

$$\begin{aligned} B_1(u) &= B_1(A_1 + B_1\, x) = A_1 - \alpha\, B_1 = \gamma_1, \\ B_2(u) &= B_2(A_1 + B_1\, x) = A_1 + B_1 - \beta B_1 = \gamma_2 \end{aligned}$$

Then we find that

$$A_1 = \frac{\alpha\,\gamma_2 + (1-\beta)\,\gamma_1}{\Delta}, \quad B_1 = \frac{\gamma_2 - \gamma_1}{\Delta} \tag{3}$$

Finally the solution is:

$$\begin{aligned} u &= \frac{1}{\Delta}\Bigg(\alpha\,\gamma_2 + (1-\beta)\,\gamma_1 + (\gamma_2 - \gamma_1)\, x + (\beta - 1 + x) \int_0^x (\alpha + y)\, f(y)\, dy \\ &\quad + (\alpha + x) \int_x^1 (\beta - 1 + y)\, f(y)\, dy \Bigg) \end{aligned}$$

2[40]. *Solutions in the neighborhood of a regular singular point*: Determine the two values of the constant α for which it is true that all solutions of the equation $z\,u'' + (z-1)\,u' - \alpha\, u = 0$ are regular at $z = 0$, and obtain the general solution in each of these cases.

SOLUTION: $z\,u'' + (z-1)\,u' - \alpha\, u = 0$. $z = 0$ is a regular singular point of the DE. Since $p(z) = \frac{z-1}{z}$ and $q(z) = -\frac{\alpha}{z}$ then $A(z) = -1 + z \to a_0 = -1$ and $b_0 = 0$. This leads to the indical equation $\Lambda_0(r) = r(r-1) + a_0 r + b_0 =$

$r(r-2)$. Hence $r_1 = 2$ and $r_2 = 0$ which means that $r_1 - r_2 = 2$. If the recursion relations are not consistent the second solution corresponding to $r = r_2 = 0$ has logarithmic singularity. To have regular solutions we must have consistent recursion relations. Hence assuming

$$u(r,z) = \sum_{n=0}^{\infty} C_n(r) z^{n+r}$$

Inserting this into the DE we get

$$r(r-2)C_0\, z^{r-1} + \sum_{n=0}^{\infty} \left[(n+r+1)(n+r-1)\, C_{n+1}(r) + (n+r-\alpha)\, C_n(r)\right] z^{n+r} = 0 \tag{4}$$

Hence the recursion relations are

$$(n+r+1)(n+r-1)\, C_{n+1}(r) = -(n+r-\alpha)\, C_n(r), \quad n = 0, 1, 2, \cdots \tag{5}$$

For $r = r_2 = 0$ the above recursion relations are consistent when $\alpha = 1$ and $\alpha = 0$.

$$\begin{aligned}
&(n+1)(n-1)\, C_{n+1} + (n-\alpha)\, C_n = 0, \quad n = 0, 1, 2, \cdots \\
&n = 0: \qquad -C_1 - \alpha\, C_0 = 0, \\
&n = 1: \qquad (1-\alpha)\, C_1 = 0, \\
&n = 2: \qquad 2C_3 + (2-\alpha)\, C_2 = 0, \\
&\cdots\cdots\cdots
\end{aligned}$$

For consistency either $\alpha = 1$ or $C_1 = 0$ but this leads to $\alpha = 0$ (C_0 can not be zero).

(i). $\alpha = 1$.

$$(n+r+1)(n+r-1)\, C_{n+1}(r) + (n+r-1)\, C_n(r) = 0, \quad n = 0, 1, 2, \cdots \tag{6}$$

or

$$(n+r-1)[(n+r+1)\, C_{n+1}(r) + C_n(r)] = 0, \quad n = 0, 1, 2, \cdots \tag{7}$$

For $r = r_1 = 2$ we get $C_{n+1} = -\frac{C_n}{n+3}$ for all $n = 0, 1, 2, \cdots$. Then $C_n = \frac{2(-1)^n\, C_0}{(n+2)!}$ for all $n = 1, 2, \cdots$. This leads to (we took $2C_0 = 1$)

$$u_1(z) = \sum_{n=0}^{\infty} \frac{(-1)^n\, z^{n+2}}{(n+2)!} = e^{-z} + -1 + z \tag{8}$$

For $r = r_2 = 0$ we have

$$\begin{aligned}
&(n-1)[(n+1)C_{n+1} + C_n] = 0, \text{for all } \; n = 0, 1, 2, \cdots \\
&n = 0: \quad C_1 = C_0, \\
&n = 1: \quad 0 = 0, \\
&n - 2: \quad 3C_3 + 2C_2 = 0, \\
&\cdots\cdots\cdots
\end{aligned}$$

Choosing $C_2 = 0$ we get $C_n = 0$ for all $n \geq 2$. Then the solution is

$$u_2 = 1 - z \tag{9}$$

Hence the general solution corresponding to $\alpha = 1$ is given by

$$u = a\, e^{-z} + b(1 - z) \tag{10}$$

where a and b are arbitrary constants.

(ii). $\alpha = 0$.

$$(n + r + 1)(n + r - 1)\, C_{n+1}(r) + (n + r)\, C_n(r) = 0, \quad n = 0, 1, 2, \cdots \tag{11}$$

For $r = r_1 = 2$ we get

$$(n + 3)(n + 1)\, C_{n+1} + (n + 2)\, C_n = 0, \quad \text{for all } \; n = 0, 1, 2, \cdots \tag{12}$$

we find that

$$C_1 = -\frac{2}{3}\, C_0, \; C_2 = \frac{1}{4}, \; C_3 = -\frac{1}{3 \cdot 5}, \cdots \tag{13}$$

which gives

$$\begin{aligned} u_1(z) &= z^2\,(1-\frac{2}{3}z+\frac{1}{4}z^2-\frac{1}{3\cdot 5}z^2+\cdots), \\ &= -2(-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{8}+\frac{z^5}{2\cdot 3\cdot 5}+\cdots) \\ &= -2((z+1)e^{-z}-1) \end{aligned}$$

For $r = r_2 = 0$ we get

$$\begin{aligned} &(n+)(n-1)\,C_{n+1}+n\,C_n=0, \quad \text{for all} \;\; n=0,1,2,\cdots \\ &n=0: \quad C_1=0, \\ &n=1: \quad 0=0, \\ &n=2: \quad 3C_3+2C_2=0, \\ &\cdots\cdots\cdots, \end{aligned} \tag{14}$$

(15)

Choosing $C_2 = 0$ we $C_n = 0$ for all $n \geq 1$. Then the second solution $u_2(z)$ is just a constant. Hence the general solution is given by

$$u(z) = a(z+1)e^{-z} + b \tag{16}$$

where a and b are arbitrary constants.

3[40].*Existence and uniqueness of solutions in the neighborhood of an ordinary point*: Let $u'' + p(z)\,u' + q(z)u = 0$ where $p(z)$ and $q(z)$ are analytic functions in a neighborhood of a point $z = z_0$ (an ordinary point of the DE) with the initial conditions $u(z_0) = u_0$ and $u'(z_0) = u_1$. Here u_0 and u_1 are given numbers. Prove that the solution of the differential equation exists and unique.

SOLUTION: See the lecture notes.