MATH 543

METHODS OF APPLIED MATHEMATICS I
First Midterm Exam

October 31, 2013
Thursday 10.30-12.20

PROBLEMS:Choose any two of the following three questions

1[60]. **(a)** Assume that there exists an orthonormal basis $|e_i>$, $(i = 1, 2, \cdots)$ in $L^2_w(a, b)$. Then , for any $|f> \in L^2_{a,b}$, the sequence of vectors

$$|f_k> = \sum_{i=1}^{k} f^i \, |e_i>$$

with

$$f^i = <e_i|f>, \quad i = 1, 2, \cdots$$

has $|f>$ as the limit vector in the sense that

$$\lim_{k\to\infty} \rho(|f>, |f_k>) = 0.$$

SOLUTION: See lecture Notes (b) Prove that any orthogonal family $|e_i>, \quad i = 1, 2, = \cdots$ in $L^2_w(a, b)$ is linearly independent. **SOLUTION:** Assume that they are linearly dependent, i.e.,

$$\sum_{i=0} a_i \, |e_i> = 0 \tag{1}$$

where a_i are some non-vanishing constants. Taking inner product of both sides with $|e_k>$ and using the orthonormality of $|e_i>$ we get $a_k = 0$ for all $k = 0, 1, \cdots$. Contradiction. **(c)** If $(e_1, e_2, \cdots, e_n)$ is a finite orthonormal family of functions in $L^2_w(a, b)$ and $f \in L^2_w(a, b)$, then

$$|||f> - \sum_{k=1}^{n} f^k \, |e_k>||$$

has its minimum value when $f^i =< f|e_i >, \quad i = 1, 2, \cdots$ and

$$\sum_{k=1}^{n} | < f|e_k > |^2 \leq ||f||^2.$$

SOLUTION: Since e_n's are in $L^2_w(a, b)$ then

$$|||f > -\sum_{k=1}^{n} f^k \, |e_k > ||^2 = ||f||^2 - \sum_{i=1}^{n} | < e_i|f > |^2 + \sum_{i=1}^{n} |f^i - < e_i|f > |^2 \tag{2}$$

It is clear that

$$||f - \sum_{k=1}^{n} f^k \, |e_k > ||^2 \geq ||f||^2 - \sum_{i=1}^{n} | < e_i|f > |^2 \tag{3}$$

Equality gives the minimum value for which case the last term in (2) vanishes, i.e., $f^i =< f|e_i >, \quad i = 1, 2, \cdots$. In this case we know that the Bessel inequality $\sum_{k=1}^{n} | < f|e_k > |^2 \leq ||f||^2$ works.
2[60]. **(a)** Let $e_n(x)$, $(n = 0, 1, 2, \cdots)$ be one of the classical orthonormal polynomials, i.e., $e_n(x) = \frac{C_n(x)}{||C_n(x)||}$, with weight function $w(x)$ and $x \in [a, b]$. Prove that the sequence

$$h_n(x) = w(x) \sum_{k=0}^{n} e_k(x) \, e_k(0)$$

where $0 \in [a, b]$ defines the delta function $\delta(x)$. [**Hint**: You have to prove (a) $\int_a^b \delta(x) \, dx = 1$, and (b) $\int_a^b \delta(x) g(x) \, dx = g(0)$ for any test function $g(x)$, and take $e_0(x) = 1$]
SOLUTION: (i) Since $< e_i|e_j >= \delta_{ij}$ then

$$\int_a^b h_x dx = \sum_{k=0} e_k(0) \int_a^b w(x) e_k(x) dx = \sum_{k=0} e_k(0) \, \delta_{k0} = e_0(0) = 1 \tag{4}$$

(ii) For any good function $g(x)$ we have

$$\lim_{n\to\infty} \int_a^b h_n(x) \, g(x) dx = \lim_{n\to\infty} \sum_{k=0}^{n} e_k(0) \int_a^b w(x) e_k(x) \, g(x) dx \tag{5}$$

Since $g \in L^2(a,b)$ then $a_k = \int_a^b w(x) e_k(x)\, g(x) dx$, Fourier coefficients of the function g, and $g(x) = \sum_{k=0}^{\infty} a_k\, e_k(x)$. Then (5) reduces to

$$\lim_{n\to\infty} \int_a^b h_n(x)\, g(x) dx = \lim_{n\to\infty} \sum_{k=0}^{n} a_a\, e_k(0) dx = g(0) \tag{6}$$

(b) Consider the following sequence:

$$h_n(x) = \begin{cases} 0 & \text{if } x \le \frac{-1}{n} \\ \frac{(nx+1)}{2} & \text{if } \frac{-1}{n} \le x \le \frac{1}{n} \\ 1 & \text{if } x \ge \frac{1}{n} \end{cases}$$

Prove $\frac{dh_n(x)}{dx} \to \delta(x)$. Hence formally we may write that $\frac{d\theta(x)}{dx} = \delta(x)$. **SOLUTION: See the Lecture Notes**

3[60]. Let $H_n(x)$, $(n = 0, 1, 2, \cdots)$, $x \in (-\infty, \infty)$ with the weight function $w = e^{-x^2}$ be the Hermite polynomials where the Rodriguez formula is given by

$$H_n(x) = (-1)^n\, e^{x^2}\, \frac{d^n}{dx^n} e^{-x^2}, \quad n = 0, 1, 2, \cdots$$

Prove the following: **(a)** H_n's are polynomials of degree n.**SOLUTION**: Use proof by induction. $H_1(x)$ is clearly a polynomial of order 1. Assume that $H_n(x)$ is a polynomial of order n. Then

$$H_{n+1}(x) = -(-1)^n\, e^{x^2}\, \frac{d^{n+1}}{dx^{n+1}} e^{-x^2} = -\frac{d}{dx}\left[(-1)^n\, e^{x^2}\, \frac{d^n}{dx^n} e^{-x^2}\right] + 2x\, H_n(x)$$

which is clearly a polynomial of order $n+1$. **(b)** H_n's are orthogonal to any polynomial of degree $m < n$. **See Lecture Notes (c)** $H_n(x)$ has exactly n distinct real roots in $\mathbb{R}$. **See Lecture Notes**