MATH 543
METHODS OF APPLIED MATHEMATICS I
Final Exam

January 8, 2012
Tuesday 12:30-14:30 (SAZ02)

PROBLEMS: Choose any three of the following four problems.

1[35]. Hypergeometric equation is given as

$$z(1-z)\,u'' + [c - (1+a+b)\,z]\,u' - ab\,u = 0.$$

One of the solution of this equation about $z = 0$ is given by

$$u = F(a,b,c,z) = \sum_{n=0}^{\infty} \frac{a_n\, b_n}{c_n}\, \frac{z^n}{n!}$$

where

$$a_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad b_n = \frac{\Gamma(b+n)}{\Gamma(b)}, \quad c_n = \frac{\Gamma(c+n)}{\Gamma(c)} \quad |z| \le 1$$

(a). Prove that the general solution of the hypergeometric equation about its regular singular point at $z = \infty$, provided $1-c$, $b-a$, and $c-a-b$ are not integers, is given by

$$u(z) = A\, z^{-a}\, F(a, a-c+1; a-b+1; \frac{1}{z}) + B\, z^{-b} F(b, b-c+1; b-a+1; \frac{1}{z}),$$

where A and B are arbitrary constants.

b. Prove that

$$\frac{d^n}{dz^n}\, F(a,b;c;z) = \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)}\, F(a+n, b+n; c+n; z)$$

for all $n = 0, 1, 2, \cdots$

2[35]. Let

$$y'' + 2y' + e^{\varepsilon y} = 1, \quad x > 0, \ 0 < \varepsilon << 1 \tag{1}$$
$$y(0) = 1, \quad y(\infty) = 0 \tag{2}$$

Find a uniformly valid first order approximation of this boundary value problem.

3[35]. Solve the following boundary value problem by using the method of Green's function

$$u''' + u'' = f(x), \quad x > 0$$
$$u(0) = u'(0) = u''(0) = 0.$$

4[35]. Let

$$y'' + y = \varepsilon\, y(y')^2, \quad t > 0$$

where $y(0) = 1$ and $y'(0) = 0$ and $0 < \varepsilon << 1$. Find a two term (uniformly valid) perturbation approximation. Discuss the convergence of the perturbation series (whether the approximate solution is a uniformly valid one).