MATH 543
METHODS OF APPLIED MATHEMATICS I
Second Midterm Exam

December 11, 2012
Tuesday 13.40-15.30
Solutions

PROBLEMS: Choose any three of the following four problems.

1[35]. Let $(xu')' - \frac{n^2}{x}u = x^2$. Boundary conditions are such that $u(1) = 0$ and $u(0)$ is finite. Here n is a positive real number and $f(x)$ is a given function. Solve this boundary value problem by using the method Green's function.

2[35]. Use the method of Frobenius to obtain the general solution of the equation, valid near $z = 0$:

$$zy'' + 2y' + zy = 0$$

3[35]. A boundary value problem is given as follows

$$\begin{aligned} u''' - u'' + u' + x^2u = f(x), &\quad a < x < b, \\ u(a) - \alpha u'(a) = 0, & \\ u(b) - \beta u'(b) = 0 & \\ u''(a) = 0, & \end{aligned} \tag{1}$$

where α and β are real constants. Find the adjoint boundary value problem.

Lecture 9-10

4[35]. Let $z = z_0$ be a regular singular point of a second order differential equation $L(u) = u'' + p(z)z' + q(z)u = 0$ where $p(z)$ and $q(z)$ are continuous functions except the point $z = z_0$. Suppose that the roots of the r_1 and r_2 of the indical equation satisfy $r_1 > r_2$ and differ by a positive integer N, i.e., $r_1 - r_2 = N \neq 0$ and $C_n(r)$ satisfy recursion relation when

$$u(z) = (z - z_0)^r \sum_{n=0}^{\infty} C_n(r)\,(z - z_0)^n.$$

1) $(xu')' - n^2 u = x^3$

soln. of the hom. eqn

$x = e^{\alpha}$ $\quad u' = u_\alpha \frac{1}{x}$

$\Rightarrow u_{\alpha\alpha} - n^2 u = 0 \qquad u = a\, x^n + b\, x^{-n}$

$u_1 = x^n, \quad u_2 = x^{-n}$

$$G(x,y) = \begin{cases} a_1 x^n + b_1 x^{-n}, & 0 \le x \le \xi \\ a_2 x^n + b_2 x^{-n}, & \xi \le x < 1 \end{cases}$$

- $G(0,y)$ is finite $\Rightarrow b_1 = 0$
- $G(1,y) = 0 \qquad a_2 + b_2 = 0$
- continuity:

$a_1 y^n = a_2 y^n + b_2 y^{-n} = a_2 (y^n - y^{-n})$

$a_1 = a_2 (1 - y^{-2n})$

- Jump.

$x\, G'(x,y) \Big|_-^+ = \cancel{y}\, 1$

$\cancel{y}(G'_> - G'_<) = \cancel{y} \Rightarrow G'_> - G'_< = 1/y$

$n a_2 \cancel{y}^{n-1} \cancel{+} n b_2 \cancel{y}^{-n-1} - a_1 n y^{n-1} = \cancel{1}/y$

$\cancel{a_2 - b_2 y^{-2n} = a_1 y^{1-n}}$

$\cancel{a_2 (1 + y^{-2n}) = a_1 y^{1-n} = a_1}$

$$n\, a_2 \left(y^{n-1} + y^{-n-1} \right) - a_1 n y^{n-1} = 1/y$$

(2)

$$a_2 = \frac{1/y}{2n y^{-n-1}} = \frac{y^n}{2n}$$

$$a_1 = \frac{1}{2n} y^n \left(1 - y^{-2n}\right) \quad , \quad b_1 = -a_2$$

$$\Rightarrow \quad G(x,y) = \begin{cases} \frac{1}{2n} \left[(xy)^n - (x/y)^n \right] & x \le y \\ \frac{1}{2n} \left[(xy)^n - (y/x)^n \right] & x \geq y \end{cases}$$

$$u(x) = \int_0^1 G(x,y) f(y)\, dy \quad , \quad f(y) = y^2$$

$$= \int_0^x \left(\frac{y^n x^n}{2n} - \frac{y^n x^{-n}}{2n} \right) y^2 dy + \int_x^1 \left(\frac{y^n x^n}{2n} - \frac{y^{-n} x^n}{2n} \right) y^2 dy$$

$$= \frac{x^n - x^3}{n^2 - 9} \qquad n \neq 3$$

$$= \frac{1}{6} x^3 \ln x \qquad n = 3$$

(13)

(2) $zy'' + 2y' + zy = 0$

a) First way

$z=0$ is a regular singular point.

$p(t) = 2/z \Rightarrow a_0 = 2$

$q(t) = 1 \Rightarrow b_0 = 0$

$\lambda_0(r) = r(r-1) + 2r = r(r+1) = 0 \qquad r_1 = 0,\ r_2 = -1$

Hence $r_1 - r_2 = 1$. We have to check the consistency of the recursion relations

$$y(t) = \sum_{n=0}^{\infty} c_n(r) z^{n+r}$$

$$\sum_{n=0} (n+r)(r+n-1) c_n(r) z^{n+r-1} + 2\sum (n+r) c_n(r) z^{n+r-1} + \sum c_n(r) z^{n+r+1} = 0$$

$\Rightarrow$

$~~r(r-1) c_0 z + (r+n)r~~$

$$\sum_{n=0} (n+r)(n+r+1) c_n(r) z^{n-1} + \sum_{n=0} c_n(r) z^{n+1} = 0$$

$$= r(r+1) c_0 \frac{1}{z} + (r+1)(r+2) c_1 + \sum_{n=2} (n+r)(n+r+1) c_n(r) z^{n-1} + \sum_{n=0} c_n(r) z^{n+1} = 0$$

$$= r(r+1) c_0 + (r+1)(r+2) c_1 + \sum_{n=0}^{\infty} \left[(n+r+2)(n+r+3) c_{n+2} + c_n\right] z^{n+1} = 0$$

$$c_{n+2} = -\frac{c_n}{(n+r+2)(n+r+3)}, \qquad n \geq 0$$

recursion relations are consistent for all values of r. We can determine all c_i's from

$$(n+r+2)(n+r+3) c_{n+2} + c_n = 0 \qquad \forall\ n \geq 0$$

a) for the first index : $r=r_1=0$

$$c_{n+2} = -\frac{c_n}{(n+2)(n+3)}, \quad c_1 = 0$$

$$c_n = -\frac{c_{n-2}}{n(n+1)}$$

$$\Rightarrow \quad c_n = (-1)^n \frac{c_0}{(n+1)!} \qquad n = \text{even}$$

$$y_1(z) = \sum_{n=\text{even}} (-1)^{n/2} \frac{z^n}{(n+1)!} = \sum_{k=0} (-1)^k \frac{z^{2k}}{(2k+1)!}$$

$$= \frac{1}{z} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} = \frac{1}{z} \sin z$$

b) For the second index : $r=r_2=-1$ (both c_0 and c_1 are arbitrary, but we chose $c_1=0$)

$$c_{n+2} = -\frac{c_n}{(n+1)(n+2)}$$

$$c_n = -\cancel{\frac{c_{n-2}}{(n+2)(n+1)}} = -\frac{c_{n-2}}{n(n-1)}$$

$$c_n = (-1)^{n/2} \frac{1}{n!} \qquad n = \text{even}$$

$$y_2(z) = \frac{1}{z} \sum_{n=\text{even}} (-1)^{n/2} \frac{z^n}{n!} = \frac{1}{z} \sum_{k=0} (-1)^k \frac{z^{2k}}{(2k)!}$$

$$= \frac{1}{z} \cos z$$

c) general soln: $y(z) = \frac{1}{z}(a \sin z + b \cos z)$

b) second way $z=0$ is a regular singular point of y

let

$$y=\frac{1}{z}u \quad \Rightarrow \quad y'=-\frac{1}{z^2}u+\frac{1}{z}u'$$

$$y''=\frac{1}{z}u''-\frac{2}{z^2}u'+\frac{2}{z^3}u$$

$$\Rightarrow \; zy''+2y'+zy = u''-\underline{\frac{2}{z}u'}+\cancel{\frac{2}{z^2}u}-\cancel{\frac{2}{z^2}u}+\underline{\frac{2}{z}u'}+u=0$$

$$=u''+u=0 \quad \Rightarrow$$

$$u(z)=\alpha\cos z+\beta\sin z.$$

Then $\quad y(z)=\alpha\,\frac{1}{z}\cos z+\beta\,\frac{1}{z}\sin z$

SOLUTIONS of the second midterm exam

(3) $L(u) = u''' - u'' + u' + x^2 u = f(x)$, $a < x < b$

a) $L^+(v) = -v''' - v'' - v' + x^2 v = g(x)$, $a < x < b$

b) $\bar{v} L(u) - u(\overline{L^+ v}) = \frac{d}{dx} Q$

$$Q = \bar{v} u'' + u\bar{v}'' - u'\bar{v}' + u\bar{v}' - \bar{v}u' + u\bar{v}$$

Green's identity $\Rightarrow$ $Q(a) = Q(b)$

B Cs : $u(a) - \alpha u'(a) = 0$

$u(b) - \beta u'(b) = 0$

$u''(a) = 0$

$Q(a) = Q(b) \Rightarrow$

$u'(a) [\alpha \bar{v}''(a) + (\alpha-1)\bar{v}'(a) + (\alpha-1)\bar{v}(a)]$

$= u''(b)\bar{v}(b) + [\beta \bar{v}''(b) + (\beta-1)\bar{v}'(b) + (\beta-1)\bar{v}(b)] u'(b)$

$\Rightarrow$ $\alpha v''(a) + (\alpha-1)v'(a) + (\alpha-1)v(a) = 0$

$\beta v''(b) + (\beta-1)v'(b) = 0$,

$v(b) = 0$

Ad Boundary value problem is

$-v''' - v'' - v' + x^2 v = g(x)$, $a < x < b$

$\alpha v''(a) + (\alpha-1)[v'(a) + v(a)] = 0$

$\beta v''(b) + (\beta-1)v'(b) = 0$

$v(b) = 0$

$$L(u) = u''' - u'' + u' + u$$

$$L^{\dagger}(v) = -v''' - v'' - v' + u$$

$$\bar{v}\, L(u) - u\, L^{\dagger}\bar{v} = \bar{v}\,[u''' - u'' + u' + \cancel{u}] - u\,[-v''' - v'' - v' + \cancel{v}]$$

$$= \bar{v}u''' + u\bar{v}''' - \bar{v}u'' + u\bar{v}'' + \bar{v}u' + uv'$$

$$= (\bar{v}u'' + u\bar{v}'')' - (\bar{v}'u')' + (-\bar{v}u' + u\bar{v}')' + (u\bar{v})'$$

$$\Rightarrow \Phi = \bar{v}u'' + u\bar{v}'' - \bar{v}'u' + u\bar{v}' - \bar{v}u' + u\bar{v}$$

$$u(a) - \alpha u'(a) = 0, \quad u(b) - \beta u'(b) = 0, \quad u''(a) = 0$$

$$\Phi(a) = \Phi(b)$$

$$\cancel{\bar{v}(a)u''(a)}^{\;0} + u(a)\bar{v}''(a) - \bar{v}'(a)u'(a) + u(a)\bar{v}'(a) - \bar{v}(a)u'(a) + u(a)\bar{v}(a)$$

$$= \bar{v}(b)u''(b) + u(b)\bar{v}''(b) - \bar{v}'(b)u'(b) + u(b)\bar{v}'(b) - \bar{v}(b)u'(b) + u(b)\bar{v}(b)$$

$$u'(a)\,[\alpha\bar{v}''(a) - \bar{v}'(a) + \alpha\bar{v}'(a) - \bar{v}(a) + \alpha\bar{v}(a)]$$

$$= \bar{v}(b)u''(b) + [\beta\bar{v}''(b) - \bar{v}'(b) + \beta\bar{v}'(b) - \bar{v}(b) + \beta\bar{v}(b)]$$

$$\bar{v}(b) = 0, \quad \alpha\bar{v}''(a) + (-1+\alpha)\bar{v}'(a) + (\alpha-1)\bar{v}(a) = 0$$

$$\beta\bar{v}''(b) + (\beta-1)\bar{v}'(b) = 0$$

$$v(b) = 0$$