MATH 543
METHODS OF APPLIED MATHEMATICS I
Second Midterm Exam

December 11, 2012
Tuesday 13.40-15.30

PROBLEMS: Choose any three of the following four problems.

1[35]. Let $(xu')' - \frac{n^2}{x}u = x^2$. Boundary conditions are such that $u(1) = 0$ and $u(0)$ is finite. Here n is a positive real number and $f(x)$ is a given function. Solve this boundary value problem by using the method Green's function.

2[35]. Use the method of Frobenius to obtain the general solution of the equation, valid near $z = 0$:

$$zy'' + 2y' + zy = 0$$

3[35]. A boundary value problem is given as follows

$$\begin{gathered} u''' - u'' + u' + x^2 u = f(x), \quad a < x < b, \\ u(a) - \alpha u'(a) = 0, \\ u(b) - \beta u'(b) = 0 \\ u''(a) = 0, \end{gathered} \tag{1}$$

where α and β are real constants. Find the adjoint boundary value problem.

4[35]. Let $z = z_0$ be a regular singular point of a second order differential equation $L(u) = u'' + p(z)z' + q(z)u = 0$ where $p(z)$ and $q(z)$ are continuous functions except the point $z = z_0$. Suppose that the roots of the r_1 and r_2 of the indical equation satisfy $r_1 > r_2$ and differ by a positive integer N, i.e., $r_1 - r_2 = N \neq 0$ and $C_n(r)$ satisfy recursion relation when

$$u(z) = (z - z_0)^r \sum_{n=0}^{\infty} C_n(r)\,(z - z_0)^n.$$

Prove that the solution corresponding to the index $r = r_2$ (the second solution) is of the form

$$u(z) = cu_1(z)\,\ln(z - z_0) + (z - z_0)^{r_2} \sum_{n=0}^{\infty} B_n\,(z - z_0)^n$$

where u_1 is the solution corresponding to $r = r_1$ (the first solution) and c is a constant and

$$B_n = \lim_{r \to r_2} \frac{d}{dr}\,(r - r_2)C_n(r)$$