MATH 543
METHODS OF APPLIED MATHEMATICS I
First Midterm Exam

October 30, 2012
Thuesday 13.40-15.30

PROBLEMS: Choose any three of the following four problems.

1[35]. **(a)** Let $|e_i>$, $i = 1, 2, 3, \cdots$ be an infinite sequence of orthonormal vectors in $L^2_w(a, b)$. Let $f^i =< e_i|f >$, $(i = 1, 2, 3, \cdots)$ for any vector $|f >\in L^2_w(a, b)$. Show that $||f||^2 \geq \sum_{i=1}^{\infty} |f^i|^2$
(b) Let $|e_i>$ be an orthonormal basis of $L^2_w(a, b)$. Show that for any vector $|f >\in L^2_w(a, b)$ the sequence of vectors

$$|f_k >= \sum_{i=1}^{k} f^i \, |e_i >, \quad (k = 1, 2, 3, \cdots)$$

where $f^i =< e_i|f >$, $(i = 1, 2, 3, \cdots)$ has $|f>$ as the limiting vector in the sense that

$$\lim_{k\to\infty} \rho(|f >, |f_k >) = 0$$

where ρ is the distance function in $L^2_w(a, b)$.
(c) Prove that an orthonormal set of vectors $|e_i >$, $(i = 1, 2, 3, \cdots)$ form a basis of $L^2_w(a, b)$ if and only if

$$||f||^2 = \sum_{i=1}^{\infty} |f^i|^2$$

where $f^i =< e_i|f >$, $(i = 1, 2, 3, \cdots)$

2[35]. **(a)** Show that any orthogonal polynomials, $Q_n(x)$, $x \in [a, b]$ (with Q_0 =non-zero constant) in $L^2_w(a, b)$ have n distinct real roots in $[a, b]$.
(b) Let $R_n(x)$ be any polynomial in $L^2_w(a, b)$ so that the coefficient of the term x^n is unity (for all $n = 0, 1, 2, \cdots$). Show that the distance between the zero vector and R_n is minimal if R_n is proportional to $C_{(n)}(x)$. In another words classical polynomials are the only polynomials where this distance function takes its minimum value.

3[35]. Consider the following sequence:

$$h_n(x) = \begin{cases} 0 & \text{if } x \leq \frac{-1}{n} \\ \frac{(nx+1)}{2} & \text{if } \frac{-1}{n} \leq x \leq \frac{1}{n} \\ 1 & \text{if } x \geq \frac{1}{n} \end{cases}$$

(a) Prove that $h_n(x) \to \theta(x)$ where $\theta(x)$ is the step function and
(b) $\frac{dh_n(x)}{dx} \to \delta(x)$. Hence formally we may write that $\frac{d\theta(x)}{dx} = \delta(x)$

4[35]. Let $P_n(x)$, $(n = 0, 1, 2, \cdots)$, $x \in [-1, 1]$ with the weight function $w = 1$ be the Legendre polynomials where the Rodriquez formula is given by

$$P_n(x) = \frac{1}{2^n\, n!}\frac{d^n}{dx^n}(x^2 - 1)^n$$

Prove the following:
(a)

$$P'_{n+1}(x) - xP'_n(x) - (n+1)P_n(x) = 0$$

(b) Let

$$F(x,t) = \sum_{n=0}^{\infty} t^n\, P_n(x)$$

Use part (a) to show that

$$F(x,t) = \frac{1}{\sqrt{t^2 - 2xt + 1}}$$