SET 1

MATH 543: INNER PRODUCT SPACE

Some preliminaries from Haaser and Sullivan:

LINEAR SPACES

Definition 1: A linear (vector) space X over a field $\mathbf{F}$ is a set of elements together with a function , called addition, from $X \times X$ into X and a function , called scalar multiplication, from $\mathbf{F} \times X$ into X which satisfy the following conditions for all $x, y, z \in X$ and $\alpha, \beta \in \mathbf{F}$;

L.1. $(x + y) + z = x + (y + z)$
L.2. $x + y = y + z$
L.3. There is an element 0 in X such that $x + 0 = x$ for all $x \in X$.
L.4. For each $x \in X$ there is an element $-x \in X$ such that $x + (-x) = 0$.
L.5. $\alpha(x + y) = \alpha x + \alpha y$
L.6. $(\alpha + \beta)x = \alpha x + \beta x$
L.7. $\alpha(\beta x) = (\alpha\beta)x$
L.8. $1 \cdot x = x$.

Properties L.1-l.4 imply that X is an abelian group under addition and L.5-L.6 relate the opeartion of scalar multiplication to addition X and to addition and multiplication in $\mathbf{F}$.

Examples:

(a) $V_n(\mathbf{R})$. The vectors are n-tuples of real numbers and the scalars are real numbers with addition and scalar multiplication defined by

$$(\alpha^1, \cdots, \alpha^n) + (\beta^1, \cdots, \beta^n) = (\alpha^1 + \beta^1, \cdots, \alpha^n + \beta^n) \tag{1}$$
$$\beta(\alpha^1, \cdots, \alpha^n) = (\beta\alpha^1, \cdots, \beta\alpha^n) \tag{2}$$

Prove that $V_n(\mathbf{R})$ is a linear space over $\mathbf{R}$. Similarly, the set of all n-tuples of comples numbers with the above definition of addition and multiplication is a linear space over $\mathbf{C}$ and is denoted as $V_n(\mathbf{C})$.

(b). The set of all functions from a nonemty set into a field $\mathbf{F}$ with addition and scalar multiplication defined by

$$[f+g](t) = f(t) + g(t) \quad and \quad [\alpha f](t) = \alpha f(t); \quad f, g \in X,\ t \in T \tag{3}$$

is a linear space. **Prove this statement.**

Let $T = \mathbf{N}$ the set of all positive integers and X is the set of all sequences of elements $\mathbf{F}$ with addition and scalar multiplication defined by

$$(\alpha^n + (\beta^n) = (\alpha^n + \beta^n) \tag{4}$$

$$\beta(\alpha^n) = (\beta\alpha^n) \tag{5}$$

denoted as $V_\infty(\mathbf{F})$, form a linear space.

METRIC SPACES

Remember the distance function in the Euclidean space $\mathbf{R}^n$. Let $x, y, z \in \mathbf{R}^n$ then

(1) $|x-y| \geq 0$; $|x-y| = 0$ if and only if $x = y$;
(2) $|x-y| = |y-x|$;
(3) $|x-y| \leq |x-z| + z - y|$.

Definition 2: A **metric** or **distance function** on a set X is a real valued function d defined on $X \times X$ which has the following properties: for all $x, y, z \in X$.

(1) $d(x,y) \geq 0$; $d(x,y) = 0$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$;
(3) $d(x,y) \leq d(x,z) + d(z,y)$

A **metric space** (X, d) is a nonempty set X and a metric d defined on X.

Examples; In addition to the Euclidean spaces let us have the following examples. *Here all functions are assumed to be continuous* . Let $\mathcal{L}_p$ denotes a set of complex valued functions in $\mathbf{R}^n$ such that $|f|^p$ is integrable. Let us recall some results concerning such functions.

Höder's Inequality: If $p > 1, 1/q = 1 - 1/p$

$$\int |fg| \leq [\int |f|^p]^{1/p} [\int |g|^q]^{1/q}.$$

Minkowski's Inequality: If $p \geq 1$

$$[\int |f+g|^p]^{1/p} \leq [\int |f|^p]^{1/p} [\int |g|^p]^{1/p}$$

If x_k and y_k for $k = 1, \cdots, m$ are complex numbers, let $f(t) = |x_k|$ and $g(t) = |y_k|$ for $t \in [k, k+1]$ and $f(t) = 0 = g(t)$ for $t \ni [1, m+1]$. Then we obtain the summation form of the above inequalities from the integral form

Hölder's Inequality:

$$\sum_{k=1}^{m} |x_k y_k \leq [\sum_{k=1}^{m} |x_k|^p]^{1/p} [\sum_{k=1}^{m} |y_k|^q]^{1/q}$$

Minkowski's Inequality:

$$[\sum_{k=1}^{m} |x_k + y_k|^p]^{1/p} \leq [\sum_{k=1}^{m} |x_k|^p]^{1/p} + [\sum_{k=1}^{m} |y_k|^p]^{1/p}$$

Examples:

1. For $x = (\alpha^1, \cdots, \alpha^n)$ and $y = (\beta^1, \cdots, y^n)$ in $V_n(\mathbf{R})$ anf $p \geq 1$, define

$$d(x, y) = [\sum_{k=1}^{n} |\alpha^k - \beta^k|^p]^{1/p} \tag{6}$$

Show that d is a metric on $V_n(\mathbf{R})$.

2. Show that , for $p \geq 1$, $\mathcal{L}_p$ is a linear space and if

$$d(x, y) = [\int |f - g|^p]^{1/p} \tag{7}$$

then d is a metric on $\mathcal{L}_p$.

3. **Cauchy sequence**: A sequance (x_n) of points in a metric space (X, d) is called a Cauchy sequence if for each $\varepsilon > 0$ there exists a postive integer n_0 such that $d(x_n, x_m) < \varepsilon$ whenever $n, m \geq n_0$. You can prove that *a convergent sequence in metric space is Cauchy sequence*

4. A metric space X is **complete** if every Cauchy sequence in X converges to a point in X. Prove that $\mathbf{R}$ is complete with respect to the distance function $d(x, y) = |x - y|$ where $x, y \in X$. Is $\mathbf{C}$ complete with respect to the same metric.

NORMED LINEAR SPACES

Definition 3. A norm on X is a real valued function, whose value at x is denoted by $||x||$, satisfying the following conditions for all $x, y \in X$ and $\alpha \in \mathbf{F}$;

(1) $||x|| > 0$ if $x \neq 0$
(2) $||\alpha x|| = |\alpha| ||x||$
(3) $||x + y|| \leq ||x|| + ||y||$.

A linear space X with a norm defined on it is called a **normed linear space**.

Examples

1. l_p^n. On the linear space $V_n(\mathbf{F})$ define

$$||x|| = [\sum_{k=1}^{n} |\alpha^i|^p]^{1/p} \tag{8}$$

where $p \geq 1$ is nay real number and $x = (\alpha^1, \cdots, \alpha^n)$. Prove that this defines a norm (called p-norm) on $V_n(\mathbf{F})$. This space is caled l_p^n. If $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$ it a real or complex space l_p^n. In the real case if $p = 2$, then the $p - norm$

is the Euclidean norm and the space l_2^n is the Euclidean n-space $\mathbf{R}^n$. In the complex case if $p = 2$, the space l_2^n is called the unitary n space and denoted by $\mathbf{C}^n$.

2. $\mathbf{l_\infty^n}$. On the linear space $V_n(\mathbf{F})$ the infinity-norm is defined by

$$||x||_\infty = max\,\{|\alpha^i|, \cdots, \alpha^n\} \tag{9}$$

where $x = (\alpha^1, \cdots, \alpha^n)$. Prove that this defines a norm in $V_n(\mathbf{F})$. The corrseponding linear space is denoted by l_∞^n

3. $\mathbf{l_p}$. This space is a subspace of $V_\infty(\mathbf{F})$ with the norm defined by

$$||x||_p = [\sum_{k=1}^{\infty} |\alpha^i|^p]^{1/p} \tag{10}$$

where $x = (\alpha^i)$. Prove that this is a norm on $V_\infty(\mathbf{F})$. (See page 163 of HS).

4. $\mathbf{l_\infty}$. The set of all bounded sequences of scalars is also a subspace of $V_\infty(\mathbf{F})$. If on the linear space of bounded sequences of scalars we define the infinity-norm by

$$||x||_\infty = sup\,\{|\alpha^i| : i \in \mathbf{N}\} \tag{11}$$

where $x = (\alpha^i)$.

5. $\mathcal{L}_\mathbf{p}(\mathbf{E})$. For any real number $p \geq 1$ let $\mathcal{L}_p(E)$ be the set of Lebesgue measurable scalar-valued functions f defined on Lebesgue measurable set E in $\mathbf{R}^n$ such that $|f|^p$ is integrable. Let $N = \{f : f = 0 almost everywhere on E\}$, then

$$||f|| = [\int_E |f|^p]^{1/p} \tag{12}$$

defines a norm. This defines the normed linear space $\mathcal{L}_p(E)$.

6. **Cauchy property**. The sequence $\{f_n\}$ is said to have the Cauchy property uniformly on a set E if for any $\varepsilon > 0$ there is an N such that $||f_n - f_m|| < \varepsilon$ whenever $n, m > N$. This means

$$\lim_{m,n} ||f_n - f_m||_E = 0 \tag{13}$$

It is easy prove that if a sequence $\{f_n\}$ has the Cauchy property uniformly on set E then there is a function F to which this sequence converges uniformly on E.

A property often used in series of functions: If $|| \lim_{m,n\to\infty} \sum_m^n u_k||_E = 0$, then $\sum u_k$ is uniformly convergent.

INNER PRODUCT SPACES

1. **Definition**: Let X be a linear space over the field $\mathbf{F}$ where $\mathbf{F}$ is either $\mathbf{R}$ or $\mathbf{C}$. An **inner product** space on X is a scalar-valued function on $X \times X$, whose values are denoted by $< x, y >$, which has the following properties:

(1) $< x, x > \geq 0$; $< x, x >= 0$ if and only if $x = 0$;
(2) $< x, y >= \overline{< x, y >}$;
(3) $< x + y, z >=< x, z > + < y, z >$;
(4) $< \alpha x, y >= \alpha < x, y >$

where the bar in property 2 denotes the complex conjugation. A linear space with inner product defined on it is called an **inner product space**.

Prove the following properties:

(a) $< x, y + z >=< x, y > + < x, z >$
(b) $< x, \alpha y >= \bar{\alpha} < x, y >$
(c) $< x, 0 >= 0 =< 0, y >$.

2. Let X be an inner product space, and $x, y \in X$ show that

$$| < x, y > | \leq \sqrt{< x, x >}\sqrt{< y, y >} \tag{14}$$

3. Let X be an inner product space. Show that $||x|| = \sqrt{< x, x >}$ defines a norm on X.
4. Not all normed linear spaces are inner product spaces. Give examples.
5. Let X be the space of continuos complex-valued functions on $[a, b]$ with $\mathbf{F} = \mathbf{R}$ and define

$$< x, y >= \int_a^b x\,\bar{y}. \tag{15}$$

Verify that this is an inner product on X.

6. If $\sum_{k=1}^{\infty} x_k$ converges in the inner product space X, show that

$$< \sum_{k=1}^{\infty} x_k, y >= \sum_{k=1}^{\infty} < x_k, y > . \tag{16}$$

7. If X is a real inner product space and $||x|| = \sqrt{< x, x >}$, show that

$$< x, y >= \frac{1}{4}[||x + y||^2 - ||x - y||^2]. \tag{17}$$

8. If X is a complex inner product space and $||x|| = \sqrt{< x, x >}$, show that

$$< x, y >= \frac{1}{4}\,[||x + y||^2 - ||x - y||^2 + i||x + iy||^2 - i||x - iy||^2]. \tag{18}$$

9. Porve that the normed linner space l^2_∞ is not an inner product space.

10. If an inner product space is complete in its associated norm it is called a *Hilbert Space*. Prove that the space $C_2[a, b]$ of continuous real valued functions on the interval $[a, b]$ is not a Hilbert space.

11. Assuming that $\mathbf{C}^n$ is an inner product space use Problem 5 to obtain a formula for the inner product which has the norm of $\mathbf{C}^n$ as its associatd norm.

12. Let X be a complex normed linear space whose norm satisfies the parallelogram law. If we define

$$< x, y >= \frac{1}{4}\,[||x+y||^2 - ||x-y||^2 + i||x+iy|| - i||x-iy||^2], \tag{19}$$

show that this is an inner product on X.

Suggestion. To establish property 3, let $< x+y, z > - < x, z > - < y, z >= \frac{1}{4}\,\varphi(x,y,z) - \frac{i}{4}\varphi(x,y,iz)$ where

$$\begin{aligned} \varphi(x,y,z) = ||(x+z)+y||^2 - ||(x-z)+y||^2 - ||x+z||^2 + \\ ||x-z||^2 - ||y+z||^2 + ||y-z||^2. \end{aligned} \tag{20}$$

Apply the parallelogram law to each of the first two terms and obtain another expression for $\varphi(x,y,z)$. Average these two expressions and show that $\varphi(x,y,z) = 0$

To establish property 4, let $\varphi(\alpha) =< \alpha x, y > -\alpha < x, y >$. Show that $\varphi(\alpha) = 0$ where α is successively a positive integer, an integer, a rational number, i, a complex number $r+is$ where r and s are rational , and finally a complex number.

13. If X is an inner product space show that its completion $\tilde{X}$ can be made an inner product space.

14. A square matrix $H = [h_{ij}]$ with complex entries is said to be hermitian if $h_{ik} = \bar{h}_{ki}$ $(i,k = 1,\cdots,n)$. (A real Hermitian is said to be symmetric.) An Hermitian (or real symmetric) $n \times n$ matrix is called **positive definite** if

$$\sum_{i=1}^{n} \sum_{k=1}^{n} h^{ik}\, \alpha^i\, \bar{\alpha}^k > 0 \tag{21}$$

for all $(\alpha^1, \cdots, \alpha^n) \neq 0$.

(a). Let $H = [h_{ik}]$ be a positive definite Hermitian matrix. Show that

$$< x, y >= \sum_{i=1}^{n} \sum_{k=1}^{n} h_{ik} \alpha^i\, \bar{\beta}^k \tag{22}$$

where $x = (\alpha^1, \cdots, \alpha^n)$ and $y = (\beta^1, \cdots, \beta^n)$ defines an inner product on $V_n(\mathbf{C})$, the space of $n-$ tuples of complex numbers.

(b). Let $(e_1, \cdots, e_n)$ be a basis for a complex (real) $n-$ dimensional inner product space. If $x = \sum_{i=1}^{n} \alpha^i\, e_i$ and $y = \sum_{i=1}^{n} \beta^i\, e_i$, show that

$$< x, y >= \sum_{i=1}^{n} \sum_{k=1}^{n} h_{ik} \alpha^i \bar{\beta}^k \tag{23}$$

where $H = [h_{ik}]$ is a positive definite hermitian (symmetric) matrix

ORTHOGONALITY

Let x, y be in an inner product space X. These vectors are said to be orthogonal if $< x, y >= 0$. According to this definition the zero vector is orthogonal to every vector in X. If a vector is orthogonal to every vector in a set E, the x is said to be orthogonal to E. The set of all vectors orthogonal to E is called the *orthogonal complement* of E, denoted by $E^{\perp}$.

Suggested problems

1. If E is a set in an inner product space X, then $E^{\perp}$ is a closed linear subspace of X. (see page 283 of HS).
2. An orthogonal family $(x_s)_{s \in S'}$ of pints in X is linearly independent. (page 288 of HS).

This means that if $(x_1, \cdots, x_n)$ is an orthogonal n-tuple in an inner product space X of dimension n, then $(x_1, \cdots, x_n)$ is a Hamel basis, meaning that any element $x \in X$ can be expressed in the form:

$$x = \sum_{k=1}^{n} \alpha^k \, x_k \tag{24}$$

where

$$\alpha^k = \frac{< x, x_k >}{||x_k||^2} \tag{25}$$

3. Prove the following: If $(u_1, \cdots, u_n)$ is a finite orthogonal family of points in an inner product space X and $x \in X$, then

$$||x - \sum_{k=1}^{n} \alpha^k u_k||$$

has its minimum value when $\alpha^k =< x, u_k >$ and

$$\sum_{k=1}^{n} | < x, u_k > |^2 \leq ||x||^2$$

.

4. Prove the following: Let $(u_s)_{s \in S}$ be an orthonormal family of points in an inner product space X and let $x \in X$. Then, the set $\{s \in S :< x, u_s > \neq 0\}$ is complete.

5. Prove the Bessel's Inequality: Let $(u_s)_{s \in S}$ be an orthonormal family of points in iner product space X. Then for any $x \in X$ we have

$$\sum_{s \in S} | < x, u_s > |^2 \leq ||x||^2 \tag{26}$$