MATH 543

METHODS OF APPLIED MATHEMATICS I
Final Exam

January 7, 2019
Thursday 15.30-17.30
SA-Z03

SOLUTIONS

PROBLEMS: There are six questions in this exam, choose any four of them Show all your work.

[35]**(1)**. Let $y'' + 3y' + 2y = \varepsilon y^2$ for $t > 0$ and $\varepsilon << 1$ with $y(0) = 1, \quad y'(0) = -2$. Find a first order uniformly valid approximation of this initial value problem.

[35]**(2)**. Find a first order approximate solution of the following problem

$$y'' + 4y = \epsilon\, y^3$$
$$y(0) = 1, \quad y'(0) = 0,$$

where $\epsilon << 1$. Discuss the validity of the approximate solution.

[35]**(3)**.
(a) Solution of the hypergeometric equation about the regular singular point $z = 0$ with index $r_1 = 0$ is $F(a, b, c; z)$. Find the solution of the hypergeometric equation about the regular singular point $z = 0$ with index $r_2 = 1 - c$ so that $1 - c$ is not an integer or zero.
(b) Prove that

$$\frac{d^n}{dz^n} F(a, b; , c; z) = \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)} F(a+n, b+n; c+n; z)$$

for all $n = 1, 2, \cdots$
(see the appendix at the end of the questions)

[35]**(4)**. Solve the following problem by the of Green's function

$$y'' - k^2 y = f(x), \quad 0 < x < a,$$
$$y(0) = 0, \quad y(a) = 0 \tag{1}$$

where k is a positive constant. Here $f(x)$ is a given function.

[35]**(5)** Prove that the classical polynomials $C_n(x)$ where $x \in I = [a, b]$ have n-distinct roots in I.

[35]**(6)**.**(6.a)** Let

$$I(\lambda) = \int_a^b f(t)\, e^{\lambda g(t)}\, dt$$

where f is continuous and the maximum of g occurs at $t = b$ one of the end point of the interval with $g'(t) > 0$ for all $t \in [a, b]$. Then show that

$$I(\lambda) \sim \frac{f(b)\, e^{\lambda g(b)}}{\lambda g'(b)}$$

for $\lambda >> 1$. **(6.b)** Find the dominant term of

$$J(\lambda) = \int_0^{\pi/4} \cos t\, e^{\lambda \sin^2 t}\, dt$$

for $\lambda >> 1$.

Appendix: Hypergeometric equation and hypergeometric function.

$$z(1-z)\, u'' + [c - (1 + a + b)\, z)\, u' - ab\, u = 0$$

$$u = F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{a_n\, b_n}{c_n} \frac{z^n}{n!}$$

where

$$a_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad b_n = \frac{\Gamma(b+n)}{\Gamma(b)}, \quad c_n = \frac{\Gamma(c+n)}{\Gamma(c)}, \quad |z| < 1$$

Solution.

4. $y'' + 3y' + 2y = \varepsilon y^2$, $t \in [0, \infty)$

$y(0) = 1$, $y'(0) = -2$

Find a two term perturbation (first order perturbation) uniformly valid solution.

i) Let $y(t, \varepsilon) = y_0(t) + \varepsilon y_1(t) + \cdots$

$\Rightarrow y_0'' + 3y_0' + 2y_0 = 0$, $y_0(0) = 1$, $y_0'(0) = -2$

$y_1'' + 3y_1' + 2y_1 = y_0^2$, $y_1(0) = 0$, $y_1'(0) = 0$

ii) $y_0(t) = a e^{-t} + b e^{-2t}$, using the initial cond

$$a + b = 1$$
$$-a - 2b = -2$$

$a = 0$, $b = 1$

$y_0(t) = e^{-2t}$

iii) $y_1(t) = \alpha e^{-t} + \beta e^{-2t} + \gamma e^{-4t}$, $\gamma = \frac{1}{6}$

$$\left.\begin{aligned} \alpha + \beta + \gamma &= 0 \\ -\alpha - 2\beta - 4\gamma &= 0 \end{aligned}\right\} \begin{aligned} \alpha &= \tfrac{1}{3} \\ \beta &= -1/2 \end{aligned}$$

$$y_1(t) = \frac{1}{3} e^{-t} - \frac{1}{2} e^{-2t} + \frac{1}{6} e^{-4t}$$

iv) $y_{ap}(t,\varepsilon) = y_0(t) + \varepsilon y_1(t)$

$$= e^{-2t} + \varepsilon\left[\frac{1}{3}e^{-t} - \frac{1}{2}e^{-2t} + \frac{1}{6}e^{-4t}\right]$$

v) residue function $r(t,\varepsilon)$ is

$$r(t,\varepsilon) = y_{ap}'' + 3y_{ap}' + 2y_{ap} - \varepsilon y_{ap}^2$$

$$= y_0'' + \varepsilon y_1'' + 3(y_0' + \varepsilon y_1') + 2(y_0 + \varepsilon y_1)$$

$$- \varepsilon(y_0^2 + 2\varepsilon y_0 y_1 + \varepsilon^2 y_1^2)$$

$$= (y_0'' + 3y_0' + 2y_0) + (y_1'' + 3y_1' + 2y_1 - y_0^2)$$

$$- 2\varepsilon^2 y_0 y_1 - \varepsilon^3 y_1^2$$

$$= -2\varepsilon^2 y_0 y_1 - \varepsilon^3 y_1^2$$

$$r(t,\varepsilon) = -2\varepsilon^2 e^{-3t}\left[\frac{1}{3} - \frac{1}{2}e^{-t} + \frac{1}{6}e^{-3t}\right)$$

$$- \varepsilon^3 e^{-2t}\left[\frac{1}{3} - \frac{e^{-t}}{2} + \frac{1}{6}e^{-3t}\right]^2$$

$$|r(t,\varepsilon)| \le 2\varepsilon^2 + \varepsilon^3 \qquad \text{for all } t \ge 0$$

$$\Rightarrow \lim_{\varepsilon\to 0} |r(t,\varepsilon)| = 0 \quad \Rightarrow \quad \lim_{\varepsilon\to 0} r(t,\varepsilon) = 0$$

for all $t \ge 0$

Hence y_{ap} is uniformly valid solution

SOLUTION

2) $y'' + 4y = \varepsilon y^3$, $t > 0$, $\varepsilon << 1$

$y(0) = 1$, $y'(0) = 0$

We use the Poincare'-Lindstead method to find a two term approximate solution

$$y(\xi) = y_0(\xi) + \varepsilon y_1(\xi) + \cdots$$

where $\xi = \omega(\varepsilon) t$ and

$$\omega(\varepsilon) = \omega_0 + \varepsilon \omega_1 + \cdots$$

Differential equation changes to

$$\omega^2 y_{\xi\xi} + 4y = \varepsilon y^3$$

Then we use the RPM, i.e.,

$$y(\xi, \varepsilon) = y_0(\xi) + \varepsilon y_1(\xi) + \cdots$$

$$\omega(\varepsilon) = \omega_0 + \varepsilon \omega_1 + \cdots$$

$$\Rightarrow (\omega_0 + \varepsilon \omega_1 + \cdots)^2 [y_{0\xi\xi} + \varepsilon y_{1\xi\xi} + \cdots]$$

$$+ 4[y_0 + \varepsilon y_1 + \cdots] = \varepsilon (y_0 + \varepsilon y_1 + \cdots)^3$$

comparing the coefficients of ε^n, $n = 0, 1, 2, \ldots$ we get

i) $\omega_0^2 y_{0\xi\xi} + 4 y_0 = 0, \quad y_0(0) = 1, \; y_{\xi}(0) = 0$

Letting $\omega_0 = 2$ we get $y_0(\xi) = A\cos\xi + B\sin\xi$.

Using the initial conditions we obtain

$$y_0(\xi) = \cos\xi$$

ii) $\omega_0^2 y_{1\xi\xi} + 2\omega_0\omega_1 y_{0\xi\xi} + 4 y_1 = \varepsilon y_0^3$

$$y_1(0) = 0, \; y_{1\xi}(0) = 0.$$

$$\Rightarrow \quad 4 y_{1\xi\xi} + 4 y_1 = 4\omega_1 \cos\xi + \cos^3\xi$$

$$\cos^3\xi = \frac{1}{4}[\cos 3\xi + 3\cos\xi]$$

$$\Rightarrow \quad 4 y_{1\xi\xi} + 4 y_1 = 4\omega_1 \cos\xi + \frac{1}{4}[\cos 3\xi + 3\cos\xi]$$

$$= \frac{1}{4}\cos 3\xi$$

Here we took $\omega_1 = -\frac{3}{16}$ to eliminate the "secular" terms. Then

$$y_{1\xi\xi} + 4 y_1 = \frac{1}{16}\cos 3\xi$$

we find that

$$y_1(\xi) = A\cos\xi + B\sin\xi - \frac{1}{8 \cdot 16}\cos(3\xi)$$

using the initial conditions we find

$$y_1(\xi) = \frac{1}{8\cdot 16}\cos\xi - \frac{1}{8\cdot 16}\cos(3\xi)$$

Hence the approximate solution is

$$y_{app}(\xi,\varepsilon) = \cos\xi + \frac{\varepsilon}{108}\left[\cos\xi - \cos 2\xi\right]$$

$$\omega(\varepsilon) = 2 - \frac{3}{16}\varepsilon + \cdots$$

$$\xi = \left(2 - \frac{3}{16}\varepsilon + \cdots\right)t$$

Uniform validity of y_{app}: The residue function takes the form

$$r(t,\varepsilon) = \varepsilon^2\left[\frac{9}{16^2}\left(y_0'' + \varepsilon y_1''\right) - 3y_0^2 y_1\right] - 3\varepsilon^3 y_1^2 y_0 - \varepsilon^4 y_1^3$$

It can be shown that

$$|r(t,\varepsilon)| \leq \varepsilon^2 R_1 + \varepsilon^3 R_2 + \varepsilon^4 R_4$$

where R_1, R_2, R_3 are positive real numbers for instance $R_1 = \frac{9}{16^2}$, etc.. Hence

$\lim_{\varepsilon\to 0} r(t,\varepsilon) = 0 \quad \forall\, t > 0$. This means that our solution y_{app} is <u>uniformly valid</u>

3.2 [3.1 is solved in class (see the lecture Notes)]

i) $$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}$$

$$\frac{d}{dz} F = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^{n-1}}{(n-1)!}$$

$n-1 = m$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=-1} \frac{\Gamma(a+1+n)\Gamma(b+1+n)}{\Gamma(c+1+n)} \frac{z^n}{n!}$$

$$= \frac{\Gamma(c)}{\Gamma(c+1)} \cdot \frac{\Gamma(a+1)}{\Gamma(a)} \cdot \frac{\Gamma(b+1)}{\Gamma(b)} \cdot F(a+1, b+1, c+1, z)$$

$$= \frac{a_1 b_1}{c_1} F(a+1, b+1, c+1, z)$$

$$a_1 = \frac{\Gamma(a+1)}{\Gamma(a)} = \frac{a\Gamma(a)}{\Gamma(a)} = a.$$

$$b_1 = \frac{\Gamma(b+1)}{\Gamma(b)} = b.$$

$$c_1 = \frac{\Gamma(c+1)}{\Gamma(c)} = c.$$

(ii) Assume that

$$\frac{d^k}{dz^k} F = \frac{a_k b_k}{c_k} F(a+k, b+k, c+k; z)$$

where $a_k = \frac{\Gamma(a+k)}{\Gamma(a)}$, $b_k = \frac{\Gamma(b+k)}{\Gamma(b)}$, $c_k = \frac{\Gamma(c+k)}{\Gamma(c)}$

(iii) Take the derivative of the previous expression

$$\frac{d^{k+1}}{dz^{k+1}} F = \frac{a_k b_k}{c_k} \frac{d}{dz} F(a+k, b+k, c+k; z)$$

use the result of (i)

$$\frac{d^{k+1}}{dz^{k+1}} F(a,b,c;z) = \frac{a_k b_k}{c_k} \frac{(a+k)(b+k)}{(c+k)} F(a+k+1, b+k+1, c+k+1; z)$$

$$= \frac{a_{k+1} b_{k+1}}{c_{k+1}} F(a+k+1, b+k+1, c+k+1; z)$$

Hence our assumption in (ii) holds also for $k+1 \Rightarrow$ by the mathematical induction we proved that

$$\frac{d^n}{dz^n} F(a,b,c;z) = \frac{a_n b_n}{c_n} F(a+n, b+n, c+n; z)$$

SOLUTION

(4). $y'' - k^2 y = f(x)$, $0 < x < a$, $k \neq 0$

$y(0) = 0$, $y(a) = 0$

$$G(x,y) = \begin{cases} A e^{kx} + B e^{-kx} & , \quad x \leq y \\ C e^{kx} + D e^{-kx} & , \quad x \geq y \end{cases}$$

BC $\quad A + B = 0$

$$C e^{ka} + D e^{-ka} = 0$$

$$\Rightarrow \quad G(x,y) = \begin{cases} 2A \sinh kx & , \quad x \leq y \\ 2D e^{-ka} \sinh k(x-a) & , \quad x \geq y \end{cases}$$

continuity at $x = y$

$$A \sinh ky = D e^{-ka} \sinh k(y-a)$$

jump condition at $x = y$

$$2D e^{-ka} k \cosh k(y-a) - 2A k \cosh ky = 1$$

$$\Rightarrow \quad 2Dk e^{-ka} = \frac{\sinh(ky)}{\sinh(ka)} \Rightarrow 2D = \frac{1}{k} e^{ka} \frac{\sinh(ky)}{\sinh(ka)}$$

$$2A = \frac{\sinh k(y-a)}{k \sinh(ka)}$$

$$G(x,y) = \begin{cases} \dfrac{\sinh k(y-a)\sinh(kx)}{k\sinh(ka)}, & x \le y. \\ \dfrac{\sinh k(x-a)\sinh(ky)}{k\sinh(ka)}, & x \ge y. \end{cases}$$

Then

$$y(x) = \int_0^a G(x,y) f(y)\,dy$$

$$= \int_0^x G_{x\ge y}(x,y) f(y)\,dy + \int_x^y G_{x<y}(x,y) f(y)\,dy$$

$$y(x) = \frac{\sinh(kx)}{k\sinh(ka)} \int_0^x \sinh k(y-a) f(y)\,dy + \frac{\sinh k(x-a)}{k\sinh(ka)} \int_x^a \sinh(ky) f(y)\,dy.$$

$$y(x) = \frac{\sinh kx(x-a)}{k\sinh(ka)} \int_0^x \sinh(ky) f(y)\,dy + \frac{\sinh(kx)}{k\sinh(ka)} \int_x^a \sinh k(y-a) f(y)\,dy$$

$k \neq 0$

SOLUTION

Another common property of classical Polynomials (*)

(5) Prove that classical polynomials $C_n(x)$ with $x \in I \subset \mathbb{R}$ has n-distinct zeros in I.

proof: Let $P_m(x)$ $(m < n)$ be a polynomial of x then we have

$$\int_I w(x)\, C_n(x) P_m(x)\, dx = 0 \qquad m < n$$

hence we get $\int_I w(x)\, C_n(x)\, dx = 0$, $(\forall n)$ (P_0 = nonzero constant). This means that $C_n(x)$ changes its sign in I. Let us say that there are k number of zeros in I $(k < n)$. Let

$$C_n(x) = q(x)\, h_k(x), \quad h_k(x) \equiv \prod_{i=1}^{k} (x - x_i)$$

where $q(x)$ is a polynomial of degree $n-k$ which doesnot change its sign in I. It is clear that

$$\int_I w(x)\, C_n(x)\, h_k(x)\, dx = 0 \qquad \forall k < n$$

but the integrand is positive definite (or negative definite) in I. This is a contradiction. Hence $k = n$. There are n number of zero in I. To show they are distinct, let us assume that one of the roots, γ_k has multiplicity $\ell \geq 2$

$$h_n = (x - x_k)^{\ell} \prod_{i \neq k}^{n-\ell} (x - x_i) = (x - x_k)^{\ell}\, S_k(x)$$

Consider the integral

$$\int_I w(x)\, (x - x_k)^{\varepsilon}\, S_k\, C_n(x)\, dx =$$

where $\varepsilon = 0$ if ℓ = even and $\varepsilon = 1$ if ℓ = odd This integral is zero for all $\ell \geq 2$, but the integrand is positive definite in I. Once again a contradiction. Hence the only possibility $\ell = 1$ which mean that all zeros must be distinct

SOLUTION

6. a) See the Lecture Notes

b) Here $b = \pi/4$, $f(t) = \cos t$, $g(t) = \sin^2 t$

$a = 0$

$$J(\lambda) \sim \frac{\cos(\pi/4) \cdot e^{\lambda \sin^2(\pi/4)}}{\lambda \, 2 \sin(\pi/4) \cos(\pi/4)}$$

$$\sim \frac{1}{\sqrt{2}\,\lambda} e^{\lambda/2}$$