MATH 543
METHODS OF APPLIED MATHEMATICS I
Second Midterm Exam

December 4, 2015
Friday 10:40-12:30

PROBLEMS: Choose any three of the following five questions:

1[35].(a) Show that every linear second order differential operator $L = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x)$ with real coefficients is self adjoint provided that the weight function $w(x)$ is chosen as $w(x) = \frac{1}{a(x)}\, exp\left(\int^x dx'\, \frac{b(x'}{a(x'}\right)$ and $a(x) > 0$.
(b) Show also that the self adjoint operator L in a boundary value problem $L(u) = f(x), \quad x \in (a,b)$ with the boundary conditions $\alpha u(a) - u'(a) = 0$ and $\beta u(b) - u'(b) = 0$ is Hermitian(Here α and β are real constants).

2[35]. By using the method of Green's function find the solution of the following boundary value problem:

$$\begin{aligned} &u'' + u' = f(x), \qquad x \in (0,1), \\ &u(0) = 1, \quad u'(0) = -1 \end{aligned}$$

3[35]. Prove the existence and the uniqueness of the Green's function of boundary value problem

$$\begin{aligned} &L(u) = f(x), \qquad x \in (a,b), \\ &B_1(u) = 0, \quad B_2(u) = 0 \end{aligned}$$

where L is a linear second order hermitian operator (with real coefficients) and the corresponding homogenous equation has no nontrivial solutions satisfying both boundary conditions (Here B_1 and B_2 are also linear operators).

4[35]. Find the power series solution of the differential equation $x(1-x)u'' - 2u' + 2u = 0$ about $x = 0$.

5[35]. Let $z = z_0$ be regular singular point of the differential equation $u'' + p(z)\,u' + q(z)\,u = 0$ with the indices r_1 and r_2. If $r_1 - r_2 = N$ a positive integer then show that the solution corresponding to $r = r_2$ takes the form

$$u_2(z) = C\,u_1(z)\,\ln(z - z_0) + (z - z_0)^{r_2} \sum_{n=0}^{\infty} C_n\,(z - z_0)^n$$

where C is a constant and $u_1(z)$ is the solution corresponding to $r = r_1$.

1a)

$$Lu = a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu$$

$$= a\left(u'' + \frac{b}{a}u'\right) + cu \quad , \quad \text{let } \frac{b}{a} = \frac{p'}{p}$$

$$= \frac{a}{p}(pu')' + cu \qquad \text{let } \frac{a}{p} = \frac{1}{\omega}$$

$$= \frac{1}{\omega}\frac{d}{dx}\left(p\frac{du}{dx}\right) + cu$$

$L = L^\dagger$: and $\omega = \frac{p}{a} = \frac{1}{a} e^{\int^x \frac{b(x)}{a(x)} dx'}$, $a(x) > 0$

proof:

$$\omega \bar{v} L u = \bar{v}\frac{d}{dx}\left(p\frac{du}{dx}\right) + \omega c \bar{v} u$$

$$= \frac{d}{dx}\left(\bar{v}p\frac{du}{dx}\right) - p\frac{d\bar{v}}{dx}\frac{du}{dx} + \omega c\bar{v}u$$

$$= \frac{d}{dx}\left(p\bar{v}\frac{du}{dx}\right) - \frac{d}{dx}\left(pu\frac{d\bar{v}}{dx}\right)$$

$$+ u\frac{d}{dx}\left(p\frac{d\bar{v}}{dx}\right) + \omega c\bar{v}u$$

$$\omega\bar{v}Lu - \omega u\left[\frac{1}{\omega}\frac{d}{dx}\left(p\frac{d\bar{v}}{dx}\right) + c\bar{v}\right] = \frac{d}{dx}\varphi$$

$$L^\dagger = \frac{1}{\omega}\frac{d}{dx}p\frac{d}{dx} = L$$ with the weight function ω

$$\varphi = p(\bar{v}u' - u\bar{v}')$$

1b) $L^\dagger = L$, $L^\dagger$ acts on the function space V

but L acts on the function space U.

- U is the space of functions satisfying the boundary conditions
- V is the space of functions satisfying the adjoint boundary conditions

In order that L to be hermitian $U = V$

we have to check that the adjoint boundary conditions are the same as the boundary conditions

The surface term (from part a).

$$Q = p\left(\bar{v}(x) u'(x) - u(x)\bar{v}'(x)\right)\Big|_a^b$$

$$= p(b)\left[\bar{v}(b) u'(b) - u(b)\bar{v}'(b)\right]$$

$$- p(a)\left[\bar{v}(a) u'(a) - u(a)\bar{v}'(a)\right] = 0$$

$= \cancel{(p(b))}$ $\quad u'(b) = \beta u(b)$, $\quad u'(a) = \alpha u(a)$

$\Rightarrow$

$$p(b)\left[\beta\bar{v}(b) - \bar{v}'(b)\right]u(b) = p(a)\left[\alpha\bar{v}(a) - \bar{v}'(a)\right]u(a)$$

$u(a)$ and $u(b)$ can be zero and since they are independent $\Rightarrow$ their coefficient are zero

$\Rightarrow$ $\beta\bar{v}(b) = \bar{v}'(b)$, $\alpha\bar{v}(a) = \bar{v}'(a)$, α, β are real

$\Rightarrow v'(a) = \alpha v(a)$, $\bar{v}'(b) = \beta v(b)$

same as the BCs: $\Rightarrow L$ is hermitian.

2[35]. $L(u) = \dfrac{d^2u}{dx^2} + \dfrac{du}{dx} = f(x), \quad x \in (0,1)$ (1)

$$= \frac{1}{e^x}\frac{d}{dx}\left(e^x \frac{du}{dx}\right) = f(x), \quad x \in (0,1)$$

L is a self-adjoint operator with respect to the weight function $w(x) = e^x$. Linearly independent solutions $u_1(x)$ and $u_2(x)$ of the homogenous equation are

$$u_1(x) = 1, \quad u_2(x) = e^{-x} \qquad (2)$$

Boundary conditions are $u(0) = 1$, $u'(0) = -1$. Inhomogenous boundary conditions. First we solve the boundary value problem with homogeneous boundary conditions

$$u(0) = 0, \quad u'(0) = 0 \qquad (3)$$

The Green's function:

$$G(x,y) = \begin{cases} a_1 + b_1 e^{-x}, & x \le y \\ a_2 + b_2 e^{-x}, & x \ge y \end{cases}$$

Where a_1, a_2, b_1, and b_2 are y dependent functions. Using the homogeneous boundary condition (3) we get

$$a_1 + b_1 = 0, \quad -b_1 = 0$$

Hence

$$G(x,y) = \begin{cases} 0, & x \le y \\ a_2 + b_2 e^{-y}, & x \ge y \end{cases}$$

G is continuous at $x=y$

$$a_2 + b_2 e^{-y} = 0$$

and the jump condition

$$G_>' - G_<' = \frac{1}{\omega}$$

$$-b_2 e^{-y} = e^{-y} \Rightarrow b_2 = -1 \Rightarrow a_2 = e^{-y}$$

Then we have

$$G(x,y) = \begin{cases} 0 & , \quad x \le y \\ e^{-y} - e^{-x} & , \quad x \geq y \end{cases}$$

Solution of the homogeneous problem is

$$u_0(x) = \int_0^1 \omega(y)\, G(x,y)\, f(y)\, dy$$

$$= \int_0^x e^{y} (e^{-y} - e^{-x}) f(y)\, dy$$

$$= \int_0^x (1 - e^{y-x}) f(y)\, dy$$

Solution of the inhomogeneous problem is found from

$$u(x) = \alpha + \beta e^{-x} + \int_0^x (1 - e^{y-x}) f(y)\, dy$$

$$u(0) = \alpha + \beta = 1$$
$$u'(0) = -\beta = -1 \quad \Rightarrow \beta = 1, \ \alpha = 0$$

solution of the problem: $\boxed{u(x) = e^{-x} + \int_0^x (1 - e^{y-x}) f(y)\, dy}$

4 [35] $x(1-x)u'' - 2u' + 2u = 0$

$x = 0$ is a regular singular point of the DE

$$p(x) = -\frac{2}{x(1-x)}, \quad q(x) = \frac{2}{x(1-x)}$$

$$x\,p(x) = -\frac{2}{1-x}, \quad x^2 q(x) = \frac{2x}{1-x}$$

$$a_0 = -2, \quad b_0 = 0$$

$$\lambda(r) = r(r-1) + a_0 r + b_0 = r(r-1) - 2r = 0$$

$$r_1 = 3, \quad r_2 = 0$$

$$u(z,r) = \sum_{n=0}^{\infty} c_n(r)\, z^{n+r}$$

When inserted into the DE we get

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)\, c_n(r)\, z^{n+r-1}$$

$$- \sum_{n=0}^{\infty} (n+r)(n+r-1)\, c_n(r)\, z^{n+r}$$

$$- 2\sum_{n=0}^{\infty} (n+r)\, c_n(r)\, z^{n+r-1}$$

$$+ 2\sum_{n=0}^{\infty} c_n(r)\, z^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1) - 2(n+r)\right] c_n(r)\, z^{n+r-1}$$

$$+ \sum_{n=0}^{\infty} \left[-(n+r)(n+r-1) + 2\right] c_n(r)\, z^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+r)(n+r-3)\, c_n(r)\, z^{n+r-1} - \sum_{n=0} \left[(n+r)(n+r-1) - 2 \right] c_n z^{n+r} = 0$$

$$\sum_{n=0} (n+r)(n+r-3)\, c_n(r)\, z^{n+r-1} - \sum_{n=0}^{\infty} (n+r+1)(n+r-2)\, c_n z^{n+r} = 0$$

$$r(r-3)c_0 z^{r-1} + \sum_{n=1} (n+r)(n+r-3)\, c_n(r)\, z^{n+r-1}$$
$$- \sum_{n=0} (n+r+1)(n+r-2)\, c_n z^{n+r} = 0$$

$$\Rightarrow \sum_{n=0} (n+r+1)(n+r-2)\, c_{n+1}(r)\, z^{n+r}$$
$$- \sum (n+r+1)(n+r-2)\, c_n z^{n+r} = 0$$

$$\Rightarrow \sum_{n=0} (n+r+1)(n+r-2) \left[c_{n+1}(r) - c_n(r) \right] z^{n+r} = 0$$

Recursion relations

$$(n+r+1)(n+r-2)\left[c_{n+1}(r) - c_n(r) \right] = 0 \quad \forall n \geq 0$$

$r_1 = 3$

$$(n+4)(n+1)(c_{n+1} - c_n) = 0$$

$$c_{n+1} = c_n = c_0$$

$$u_1(z) = z^3 \sum_{n=0} z^n = \frac{z^3}{1-z} \qquad |z| < 1$$

$r_2 = 0$

$$(1+n)(n-2)\left[c_{n+1} - c_n\right] = 0$$

$$c_1 = c_0 \,, \quad c_2 = c_0$$

at $n=2$ recursion relation is satisfied consistently

$n=3 \qquad c_4 = c_3$

$n=4 \qquad c_5 = c_4 = c_3$

$$\Rightarrow \quad c_n = c_3 \quad \forall\, n \geq 3$$

Choosing $c_3 = 0 \quad \Rightarrow u_2(z) = c_0\,(1+z)$

The most general solution is

$$u(z) = \alpha\,(1+z) + \beta\,\frac{z^3}{1-z}$$

α and β are constants.