MATH 534: DIFFERENTIAL GEOMETRY
Midterm Exam: April 17, 2002

QUESTIONS: *Choose four out of five questions*

1. Let I be a differential Ideal. It is called closed if $dI \subset I$. Prove that the Cartan structure equations

$$\begin{aligned} \Sigma^i &= dw^i + w^i{}_j \wedge w^j, \\ \Omega^i{}_j &= dw^i{}_j + w^i{}_k \wedge w^k{}_j \end{aligned}$$

form a closed differential ideal.

Solution. It is straightforward to show that

$$d\Sigma^i = \Omega^i{}_k\, w^k - w^i{}_k\, \Sigma^k, \tag{1}$$
$$d\Omega^i{}_j = \Omega^i{}_k \wedge w^k{}_j - w^i{}_k \wedge \Omega^k{}_j \tag{2}$$

2. Let M be a differentiable manifold and $X, Y, Z, W \in T_p(M)$. Let R be the curvature tensor. Suppose that the multilinear function $R(X, Y, Z, W)$ satisfies $R(X, Y, Z, W) = R(X, Z, Y, W)$ for all vectors X, Y, Z, W. Prove that R is zero everywhere in M.

Solution. Use the cyclic identity $R(X, Y, Z, W)+R(X, Z, W, Y)+R(X, W, Y, Z) = 0$. The last term takes the form $R(X, Y, W, Z)$. It cancels the first term. Then the second term remains $R(X, Z, W, Y) = 0$. This equality is valid for all $X, Y, Z, W \in T_p(M)$ at all points $p \in M$. Hence the curvature tensor R vanishes.

3. Let D be a connection on a vector bundle E, and $p \in M$. Prove that there exists a local frame field $\mathbf{s}$ in a coordinate neighborhood of p such that the corresponding connection matrix is zero at p.

Solution. See Chern, Theorem 1.2 (Chapter 4. Connection on Vector Bundles)

4. (a) Prove that S^2 is an analytic manifold (b) Prove that S^2 can not be covered by a single chart,

Solution. (a) It was the first problem in the first homework. (b) We have to know that S^2 is a compact manifold. Assume that S^2 can be covered by a single chart. Its image under in R^2 under the homeomorphism φ is an open set in R^2. Since a continuous map preserves the compactness, the image must be a closed set. Hence this is a contradiction.

5. (Partition of unity on $\mathbf{R}^n$) Consider the covering of $\mathbf{R}^n$ by open balls B_i of radius a and with centers at the denumerable set of points $\{x_{(i)}\}$. The function ψ_i defined by

$$\begin{aligned}\psi_i &= 0, \quad (r_i \geq a)\\ \psi_i &= e^{1/({r_i}^2-a^2)} \quad (r_i < a)\end{aligned}$$

where ${r_i}^2 = \sum_k (x^k - {x^k}_{(i)})^2$ and $r_i \geq 0$, is C^∞ on $\mathbf{R}^n$ since

$$e^{1/({r_i}^2-a^2)}$$

and all its derivatives tend to zero as $r_i \to a-$. Also, $\psi_i \geq 0$ and supp ψ_i is compact. Hence , if we define $f_i(p)$ for any $p \in \mathbf{R}^n$ by

$$f_i(p) = \frac{\psi_i(p)}{\sum_k \psi_k(p)}$$

(the sum contains only a finite number of non-zero terms) , then $\{f_i\}$ is a smooth *partition of unity* on $\mathbf{R}^n$. *Find the partition of unity on* S^2.

Solution. Let $(U_\alpha, \varphi_\alpha)$, $\alpha = 1, 2$ be atlas of charts of S^2 (steorgaphic projections). Define subsets $B_{i\alpha}$ of S^2 by $B_{i\alpha} = (\varphi_\alpha)^{-1}(B_i)$ and let $\psi_{i\alpha} = \psi_i \circ \varphi_\alpha$. Then $\psi_{i\alpha} \geq 0$ and is of compact support on $B_{i\alpha}$, For any $p \in S^2$, let

$$f_{i\alpha} = \frac{\psi_{i\alpha}}{\Sigma_{r,\beta}\, \psi_{r,\beta}}$$

Then $f_{i\alpha} \geq 0$, $f_{i\alpha}$ has compact support, $\{supp f_{i\alpha}\}$ is locally finite, and $\sum f_{i\alpha} = 1$ for each $p \in S^2$. Therefore $f_{i\alpha}$ is a smooth partition of unity subordinate to $\{U_i\}$.