EXERCISES SET : - (09 April 2002) References: Our text book (bt Chern) and *Manifold Theory* by D. Martin.

DIFFERENTIABLE MANIFOLDS

1. An atlas on a set S is a family of charts $\mathcal{P} = \{(U_i, \varphi_i), i = 1, 2, \cdots\}$ such that

(i) S is the union of U_i, $(i = 1, 2, \cdots)$. i.e., $S = \bigcup\{U_i\}$
(ii) Any two charts in $\mathcal{P}$ are compatible in the sense that the overlap maps between the members of $\mathcal{P}$ are C^∞ diffeomorphism: For two charts (U_i, φ_i) and (U_j, φ_j) with $U_i \bigcap U_j \neq \phi$ we form the overlap map

$$\varphi_{ji} = \varphi_j \circ (\varphi_i)^{-1} \ : \varphi_i(U_i \bigcap U_j) \to \varphi_j(U_i \bigcap U_j)$$

and we require that $\varphi_i(U_i \bigcap U_j)$ is open and φ_{ji} is a C^∞ diffeomorphism (smooth mappings).

Two atlases $\mathcal{P}_1$ and $\mathcal{P}_2$ are equivalent if the union of these is also an atlas. A *differentiable structure* $\mathcal{D}$ on a set S is an equivalence class of atlases on S.

The union of atlases in a differentiable structure is the maximal atlas of the differentiable structure and a chart $\{U, \varphi\}$ is an admissible local chart. A *differentiable manifold* is the pair $(S, \mathcal{D})$. Let us now consider the sphere S^n in $\mathbf{R}^{n+1}$. We have the following two different differentiable structures (or two different atlases) on S^n
(a) (Steographic projections) Let $N = (0, \cdots, 0, 1)$ and $S = (0, \cdots, 0, -1)$ be the north and south poles of S^n. The steographic projections from N and S are respectively given by

$$\begin{aligned}&\varphi_+ \,|S^n - \{N\} \to \mathbf{R}^n, \text{with}\\ &\varphi_+(x^1, \cdots, x^{n+1}) = \left(\frac{x^1}{1 - x^{n+1}}, \cdots, \frac{x^n}{1 - x^{n+1}}\right)\end{aligned}$$

$$\varphi_- \,|S^n - \{S\} \to \mathbf{R}^n, \text{with}$$
$$\varphi_-(x^1, \cdots, x^{n+1}) = \left(\frac{x^1}{1+x^{n+1}}, \cdots, \frac{x^n}{1+x^{n+1}}\right)$$

Prove that overlap map

$$\varphi_- \circ (\varphi_+)^{-1} : \mathbf{R}^n - \{0\} \to \mathbf{R}^n - \{0\}$$

is given by

$$(\varphi_- \circ (\varphi_+)^{-1})(z) = \frac{z}{||z||^2}$$

$z \in \mathbf{R}^n - \{0\}$. Hence prove also that these two charts define an atlas on S^n and a differentiable structure. Hence S^n is a differentiable manifold.

(b) (Orthogonal projections) An alternative atlas for S^n is given by $2n+2$ charts defined by

$$\begin{aligned}
&(U_i{}^{\pm}, \varphi_i{}^{\pm}),\ i = 1, 2, \cdots, n+1 \quad U_i{}^{\pm} = \{x \in S^n \mid \pm\, x^i > 0\}, \\
&\varphi_i{}^{\pm} : U_i{}^{\pm} \to \{y \in \mathbf{R}^n \mid ||y|| < 1\}, \\
&\varphi_i{}^{\pm}(x^1, \cdots, x^{n+1}) = (x^1, \cdots, x^{i-1}, x^{i+1}, \cdots, x^{n+1})
\end{aligned} \tag{1}$$

Hence φ_i projects the hemisphere containing the pole $(0, \cdots, \pm, 0, \cdots, 0)$ on the open unit ball in tangent space to the sphere at the pole.

Show that the overlap map $\varphi_j{}^{\pm} \circ (\varphi_i{}^{\pm})^{-1}$ is given by

$$\begin{aligned}
&\varphi_j{}^{\pm} \circ (\varphi_i{}^{\pm})^{-1}(y^1, \cdots, y^n) = \\
&(y^1, \cdots, y^{i-1}, \pm\sqrt{1 - (y^1)^2 - \cdots - (y^{i-1})^2 - (y^{i+1})^2 - \cdots - (y^n)^2}, y^{i+1}, \cdots, y^n)
\end{aligned} \tag{2}$$

Show that this atlas defines also a differentiable structure on S^n.

(c) Prove that the two differentiable structure defined in (a) and (b) above are isomorphic (they are equivalent). For the proof use the definition of diffeomorphism between smooth manifolds in Chern et all page 8.

2. (A) *Product Manifolds*: See Chern et al page 8 and page 6.
(B) *Division Algebras*: An algebra with a unit element is called a division algebra A. If for each non-zero element $a \in \mathcal{A}$ there exists its inverse $a^{-1} \in \mathcal{A}$, i.e., $a \cdot a^{-1} = a^{-1} \cdot a = 1$. The only possible associative division algebras are the real, complex and quaternion numbers. If associativity is dropped , octonions should be added to the above algebras. Let N denote the norm of $\mathcal{A}$ with $N(ab) = N(a)N(b), \quad N(a) > 0$ for all $a, b \in \mathcal{A}$. If $N(a) = 0$ then $a = 0$. Topologically unit norm real numbers are $S^0 \equiv \mathbf{Z_2}$, unit norm complex numbers are $S^{-1} \equiv U(1)$, unit norm quaternions are $S^3 \equiv SU(2)$, unit norm octonions are S^7. S^7 is not a group manifold because octonions are not associative.
(C) *Quaternions*: The quaternion $\mathbf{H}$ algebra is a four dimensional algebra over $\mathbf{R}$ with bases $\{e_0 = 1, e_1, e_2, e_3\}$ satisfying $e_i \cdot e_j = -\delta_{ij} + \epsilon_{ijk} e_k$. Here we assume that there is a summation over the repeated indexes from 0 to 3. A quaternion u can be written as $u = u^i e_i$ and its conjugate $\bar{u}$ is defined by $\bar{u} = u^0 - u^1 e_1 - u^2 e_2 - u^3 e_3$. The norm $N(u)$ of u is defined by $N(u)^2 = u \cdot \bar{u} = (u^0)^2 + (u^1)^3 + (u^2)^2 + (u^3)^2$. Hence the inverse of u is given by $u^{-1} = \frac{\bar{u}}{N(u)}$. A realization of of quaternions is given by the Pauli matrices. The basis e_i is expressed as $\{e_0 = 1, \ e_k == -i\, \sigma_k\}$ where σ_k's are the Pauli sigma matrices. Then a quaternion u in $\mathbf{H}$ is given by

$$u = u^0 - i \sum_{k=1}^{3} u^k\, \sigma_k = \begin{pmatrix} u^0 - iu^3 & -u^2 - iu^1 \\ u^2 - iu^1 & u^0 + iu^3 \end{pmatrix}$$

(D) *Milnor's exotic sphere* Σ^7: See Chern et al page 6, Example 4. Take mapping τ defined in this example as

$$\tau(u, v) = \left(\frac{u}{N(u)^2}, \frac{u^h v u^j}{N(u)} \right)$$

where $h = \frac{k+1}{2}$, $j = \frac{1-k}{2}$ and k is an odd integer. Prove that Σ^7_k is differentiable (smooth) manifold. Here $k = \pm 1$ gives the standard differential structure (isomorphic) but all the others are not isomorphic to standard differential structures. Hence Σ^7 is homeomorphic but not diffeomorphic to S^7. See the original paper of Milnor , *Ann. Math.*, **64**, 399, (1956) (For differentiable structures on the sphere, see also J Milnor, *Am. J. Math.*, **81**, 962-972 (1959))

3. Prove that the real projective space RP^n is an analytic manifold. See Chern et al page 6 , Example 3.

4. (a)Prove that $\mathbf{R}$ is diffeomorphic to an open interval $(-1, 1)$ of $\mathbf{R}$.(b) Prove that the open disc is diffeomorphic to $\mathbf{R}^2$.

5. Show that the functions

$$f_n(x) = \frac{h(x-n)}{\sum_{m=-\infty}^{\infty} h(x-m)}$$

where

$$h(x) = \begin{cases} \exp\{-(1-x^2)^{-1}\} & (|x| < 1) \\ 0 & (|x| \geq 1) \end{cases}$$

form a partition of unity subordinate to the open covering $\{(n-2), (n+2)\}$ $(n \in \mathbf{Z})$ of $\mathbf{R}$.

6. Prove that the sphere S^n cannot be covered by a single chart. (You may assume that S^n is compact.

7. Prove that $CP(1)$ and S^2 have the same differentiable structure and are diffeomorphic to each other.

8.Suppose that X, Y, Z are smooth tangent vector fields on M, and $f, g \in C^\infty(M)$. Then
i) $[X, Y] = -[Y, X]$,
ii) $[X + Y, Z] = [X, Y] + [Y, Z]$,

iii) $[fX, gY] = f(Xg)Y - g(Yf)X + fg[X,Y]$,
iv) $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ with $[X,Y] = XY - YX$

9. Suppose X is a smooth tangent vector field on a manifold M. If $X_p \neq 0$ at a point $p \in M$, then there exists a local coordinate system (W, w^i) such that

$$X|_W = \frac{\partial}{\partial w^i}$$

TENSORS

1. Suppose $f : V \to W$ is a linear map. Then f^* coomutes with the exterior product.

2. A necessary and sufficient condition for the vectors $v_1, \cdots, v_r \in V$ to be linearly independent is

$$v_1 \wedge \cdots \wedge v_r = 0$$

3. Let $\mathbf{T}$ be acovariant tensor of rank s. A symmetric covariant tensor S(T) and and antisymmetric covariant tensor A(T) can be defined by

$$\begin{aligned} S(T) &= \frac{1}{s!}\sum_{\pi} \mathbf{T}(v_{\pi(1)}, \cdots, v_{\pi(s)}), \\ A(T) &= \frac{1}{s!}\sum_{\pi} sgn\,\pi\, \mathbf{T}(v_{\pi(1)}, \cdots, v_{\pi(s)}) \end{aligned} \tag{3}$$

Let $\alpha and \beta$ be covariant tensors. Then prove that

$$A(A(\alpha \otimes \beta)) = A(\alpha \otimes A(\beta)) = A(\alpha \otimes \beta)$$

4. Prove that a necessary and sufficient condition that $w \wedge w = 0$, where w is a 2-form, is that w is the exterior product of two 1-forms.

EXTERIOR DIFFERENTIAL CALCULUS

1. Suppose w is a differential 1-form on a smooth manifold M, X and Y are smooth tangent vector fileds on M. Then

$$< X \wedge Y, dw >= X < Y, w > -Y < X, w > - < [X, Y], w >$$

2. Prove that a necessary and sufficient condition for the Pfaffian system of equations $w_\alpha = 0$, $(1 \leq \alpha \leq r)$ to be completely integrable is

$$dw_\alpha \equiv 0 \; mod\, (w_1, \cdots, w_r), 1 \leq \alpha \leq r$$

3. On the manifold $M = \mathbf{R}^2 - \{0, 0\}$ define the 1-form

$$w = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$$

Prove that w is locally exact (but not globally).

4. Let M be a smooth maniold of dimension m and let S be a starlike subset of a coordinate neighborhood U of M. Then every closed $p-$form w of class C^1 on S is exact on S.
(a) $M = \mathbf{R}^2$, $p = 1$, $w = P(x, y)dx + Q(x, y)dy$. $dw = 0$ implies $Q_x = P_y$ and hence $w = d\theta$ where

$$\theta = x \int_0^1 P(tx, ty)dt + y \int_0^1 Q(tx, ty)dt$$

(b) $M = \mathbf{R}^3$, $p = 2$, $w = P(x, y, z)dy \wedge dz + Q(x, y, z)dz \wedge dx + R(x, y, z)dx \wedge dy$. $dw = 0$ implies $P_x + Q_y + R_z = 0$ and hence $w = d\theta$ where

$$\begin{aligned} \theta &= \left(\int_0^1 tP(tx, ty, tz)dt\right)(ydz - zdy) + \left(\int_0^1 tQ(tx, ty, tz)dt\right)(zdx - xdz) \\ &\quad + \left(\int_0^1 tR(tx, ty, tz)dt\right)(xdy - ydx) \end{aligned} \tag{4}$$

(c) To find the $(p-1)-$ form θ define a linear mapping $H_p : \Lambda^p(M) \to \Lambda^{p-1}(M)$ with the property that

$$H_{p-1}(dw) + d(H_p(w)) = w$$

Since $dw = 0$, then $\theta = H_p(w)$ where H_p is defined as follows. Let

$$\Omega = a(x^1, \ldots, x^m)\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

Then

$$H_p\Omega = (\int_0^1 a(tx^1, \cdots, tx^m)\, t^{p-1} dt)\sigma$$

where

$$\begin{aligned} \sigma &= x^{i_1} dx^{i_2} \wedge \cdots \wedge dx^{i_r} - x^{i_2} dx^{i_1} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i^p} + \\ & \cdots + (-1)^{p-1} x^{i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}} \qquad (5) \\ &= \Sigma_{j=1}^{p} (-1)^{p-1} x^{i_j}\, dx^{i_1} \wedge \cdots \wedge \hat{dx^{i_j}} \wedge \cdots \wedge dx^{i_p} \qquad (6) \end{aligned}$$

where a hat over letter means that factor is ommited.

5. **(de Rham Cohomolgy)** Consider the set of all closed $p-$ forms on a smooth manifold M of dimension m. This set is a vector subspace of $\Lambda^p(M)$ and denoted by $Z^p(M)$. The set of all exact $p =$ forms on M is anothe vector subspace B^p of $\Lambda^p(M)$, and since every exact form is closed ,$B^p(M) \subseteq Z^p(M)$.

Two closed $p-$ forms w, w' are said to be cohomologous if their difference is an exact $(p-1)-$ form. This defines an equivalence relation, $w \sim w'$. We denote the equivalence class by containing w by $[w]$. The equivalence classes are known as cohomology classes and the set of these classes is denoted by $H^p(M)$ where $R^p(M) = Z^p(M)/B^p(M)$. The zero element of $H^p(M)$ is the set of all exact $p-$ forms on M. $H^p(M)$ forms an abelian group which is called the pth (De Rham) cohomology group of M. The dimension of $H^p(M)$ is topological invariant called the Betti number B_p of M. If two manifolds are homeomorphic their Betti numbers are the same. The Euler characteristic χ and the Betti number are related as $\chi = \sum_{p=0}^{m} (-1)^p B_p$.
(a) Let M be a smooth manifold and contractable to a point (simply conected) Prove that $H^p(M) = 0$ $(p > 0)$. Hence $H^p(R^n) = 0$ $(p > 0)$.

(b) For any connected smooth manifold M of dimension m show that $H^0(M) = \mathbf{R}$.

(c) Prove that $H^1(S^1) = \mathbf{R}$.

6. Prove that the De Rham cohomology groups are invariant under diffeomorphisms

CONNECTION

1. Prove the connection transformation formula (1.12) and (1.14), and also curvature tarnsformation formula (1.29)

2. Prove that a connection always exists on a vector bundle

3. Suppose D is a connection on a vector bunle E, and $p \in M$. The prove that there exists a local frame field S in a coordinate neighborhood of p such that the corrseponding connection matrix w is zero at p.

4. Suppose X, Y are two arbitrary smooth tangent vector fileds on the manifold M. Prove that

$$R(X,Y) = D_X\, D_Y - D_Y\, D_X - D_{[X,Y]}$$

5. Suppose D is a torsion free affine connection on M. Prove the Bianci identity

$$R^j_{ikl,h} + R^j_{ilh,k} + R^j_{ihk,l} = 0$$

RIEMANNIAN MANIFOLDS

1. Let M be generalized Riemannian manifold and X, Y, Z be tangent vector fields on M. Let X satisfy

$$< D_Y\, X, Z > + < Y, D_Z X >= 0$$

for all Y and Z on M and for a torsion free metric connection D on M. Let $X = X^i\, \partial_i$ on some coordinate system (U, u^i) then the above equation is

$$X_{i,j} + X_{j,i} = 0$$

where $X_i = g_{ij} X^j$. This equation is called the Killing equation and X is called the Killing vector of M.
(a) Prove that

$$X_{i,jk} = R_{hkli} X^h$$

(b) Find all Killng vectors of R^n

2. Consider two neighboruring geodesics embedded ina 1-parameter congruence of geodesics given by $x^i = x^i(u, \lambda)$, where u is an affine parameter and the parameter λ lables the geodesics of the family. Let

$$t = \frac{\partial}{\partial u}, \quad \eta = \frac{\partial}{\partial \lambda}$$

Then , $D_t t =$ by definition of a geodesic and $[t, \eta] = 0$, since partial derivatives commute (It can be shown that the latter condition means that η is the vector connecting points with the same value of u on neighbouring curves of the congruence). By the definition of the curvature operator R,

$$R(t, \eta)t = D_t D_\eta t - D_\eta D_t t - D_{[t,\eta]} t$$

Since $D_t t = 0$ and D is torsion free, i.e., $D_t\eta - D_\eta t - [t, \eta] = 0$ then $R(t, \eta)t = D_t D_t\eta$ hence

$$\frac{D^2\eta^i}{du^2} = R^i_{jkl} t^j t^k \eta^l$$

This equation is known as the geodesic deviation.

3. If M is a torsion free affine connestion space, then with respect to a normal coordinate system α^i at the point x, the connection coefficients are zero at x.

4. Prove that a torsion free affine connection is completely determined by the curvature tensor.

5. Let $X, Y, Z, W \in T(M)$. Let the multilinear function $R(X, Y, Z, W)$ satisfy the condition $R(X, Y, Z, W) + R(X, Z, Y, W) = 0$. Then the curvature tensor vanishes in M.

6. Suppose M is a connected m-dimensional Riemannian manifold that is everywhere wandering. If $m \geq 3$, then M is constant curvature space.

COMPLEX MANIFOLDS.

1. Suppose J is an almost complex structure on a manifold M. Then a necessary and sufficient condition for J to be integrable is $d = \partial + \bar{\partial}$.

2. An almost complex structure J on a manifold M is integrable if and only if $\bar{\partial}^2 = 0$.

3. The Nijenhuis (or Torsion) tensor is the tensor N of type $(1,2)$ defined in terms of J by

$$N(X,Y) = [J(X), J(Y)] - J[X, J(Y)] - J[J(X), Y] - [X,Y]$$

Where X, Y are vector fields on M. Let M be an almost complex manifold with complex structure J, which is integrable. Prove that the Nijenhuis tensor vanishes.

4. Show that the Nijenhuis tensor on an almost complex manifold has the properies: (i) N(X+Y,Z)=N(X,Z)+N(Y,Z),
(ii) N(Y,X)=-N(X,Y),
(iii) N(fX,gY)=fg N(X,Y) where X, Y, Z are vector fileds on M and $f, g \in C^\infty(M)$.

5. Show that the components of N with respect to a local coordinate system $(x^\alpha, x^{\bar{\alpha}})$ are given by

$$N^i_{jk} = J^h_j \, J^i_{k,h} - J^j_k \, J^i_{j,h} - J^i_h \, J^h_{k,j} + J^i_h \, J^h_{j,k}$$

6. Prove that the Nijenhuis tensor vanishes on any almost complex manifold M of dimension 2.*Hint*: Observe that any vector filed on M is a linear combination of X and $J(X)$, and then show that $N(X, J(X)) = 0$.

7. Suppose f is a smooth, complex-valued function on a complex manifold

M. Then a necessary and sufficient condition for f to be a holomorphic function is that $\bar{\partial} f = 0$.

8. A necassary and sufficient condition for a Hermitian manifold M to be a Kahlerian manifold is that the torsion mathrix of the Hermitian connection on M is zero.

LIE GROUPS

1. Suppose $\sigma : G \to G$ is a smooth map. Then a necessary and sufficient condition for σ to be a right transilation of the Lie group G (of dimension r) is that it preserves the right fundamental differential form, i.e.,

$$\sigma^\star w^i = w^i, \quad 1 \le i \le r$$

2. If X, Y are right-invariant vector fields on G, then $[X, Y]$ is also a right-invariant vector field on R.

3. Let w be a 1-form satisfying (the Maurer-Cartan equations)

$$dw^i = -\frac{1}{2} c^i_{jk} \, w^j \wedge w^k,$$

and $c^i_{jk} + c^i_{kj} = 0$. Here c^i_{jk}'s are constants. Prove that

$$c^i_{jk} \, c^j_{hl} + c^i_{jh} \, c^j_{lk} + c^i_{jl} \, c^j_{kh} = 0$$

4. Find the structure constants of $Gl(n, \mathbf{R})$. Let $A, B \in gl(n; \mathbf{R})$. Prove that $[A, B] = B \cdot A - A \cdot B$

5. Prove that $\mathbf{R}^n$ is an n-dimensional Lie group with respect to the addition of vectors.

6. Prove that $GL(n, \mathbf{R})$ is a Lie group.

7. Prove that $SL(n, \mathbf{R})$ is a regular submanifold of $GL(n, \mathbf{R})$

8. Let $X = \frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. Let $w = a(x,y)dx + b(x,y)dy$, $Y = \alpha(x,y)\frac{\partial}{\partial x} + \beta(x,y)\frac{\partial}{\partial y}$. Then find $L_X Y$ and $L_X w$ where a, b, α, β are differentiable functions on an open subset of $\mathbf{R}^2$.