SET 1: 18 February , 2002 - (due 17 March 2002)

MATH 534: DIFFERENTIABLE MANIFOLDS

1. An atlas on a set S is a family of charts $\mathcal{P} = \{(U_i, \varphi_i), i = 1, 2, \cdots\}$ such that

(i) S is the union of U_i, $(i = 1, 2, \cdots)$. i.e., $S = \bigcup\{U_i\}$
(ii) Any two charts in $\mathcal{P}$ are compatible in the sense that the overlap maps between the members of $\mathcal{P}$ are C^∞ diffeomorphism: For two charts (U_i, φ_i) and (U_j, φ_j) with $U_i \bigcap U_j \neq \phi$ we form the overlap map

$$\varphi_{ji} = \varphi_j \circ (\varphi_i)^{-1} \, : \varphi_i(U_i \bigcap U_j) \to \varphi_j(U_i \bigcap U_j)$$

and we require that $\varphi_i(U_i \bigcap U_j)$ is open and φ_{ji} is a C^∞ diffeomorphism (smooth mappings).

Two atlases $\mathcal{P}_1$ and $\mathcal{P}_2$ are equivalent if the union of these is also an atlas. A *differentiable structure* $\mathcal{D}$ on a set S is an equivalence class of atlases on S.

The union of atlases in a differentiable structure is the maximal atlas of the differentiable structure and a chart $\{U, \varphi\}$ is an admissible local chart. A *differentiable manifold* is the pair $(S, \mathcal{D})$. Let us now consider the sphere S^n in $\mathbf{R}^{n+1}$. We have the following two different differentiable structures (or two different atlases) on S^n
(a) (Steographic projections) Let $N = (0, \cdots, 0, 1)$ and $S = (0, \cdots, 0, -1)$ be the north and south poles of S^n. The steographic projections from N and S are respectively given by

$$\begin{aligned}
&\varphi_+ \, | S^n - \{N\} \to \mathbf{R}^n, \text{with} \\
&\varphi_+(x^1, \cdots, x^{n+1}) = \left(\frac{x^1}{1 - x^{n+1}}, \cdots, \frac{x^n}{1 - x^{n+1}} \right)
\end{aligned}$$

$$\varphi_- \,|S^n - \{S\} \to \mathbf{R}^n, \text{with}$$
$$\varphi_-(x^1, \cdots, x^{n+1}) = \left(\frac{x^1}{1+x^{n+1}}, \cdots, \frac{x^n}{1+x^{n+1}}\right)$$

Prove that overlap map

$$\varphi_- \circ (\varphi_+)^{-1} : \mathbf{R}^n - \{0\} \to \mathbf{R}^n - \{0\}$$

is given by

$$(\varphi_- \circ (\varphi_+)^{-1})(z) = \frac{z}{||z||^2}$$

$z \in \mathbf{R}^n - \{0\}$. Hence prove also that these two charts define an atlas on S^n and a differentiable structure. Hence S^n is a differentiable manifold.

(b) (Orthogonal projections) An alternative atlas for S^n is given by $2n+2$ charts defined by

$$(U_i{}^{\pm}, \varphi_i{}^{\pm}),\ i = 1, 2, \cdots, n+1 \quad U_i{}^{\pm} = \{x \in S^n \mid \pm\, x^i > 0\},$$
$$\varphi_i{}^{\pm} : U_i{}^{\pm} \to \{y \in \mathbf{R}^n \mid ||y|| < 1\},$$
$$\varphi_i{}^{\pm}(x^1, \cdots, x^{n+1}) = (x^1, \cdots, x^{i-1}, x^{i+1}, \cdots, x^{n+1}) \tag{1}$$

Hence φ_i projects the hemisphere containing the pole $(0, \cdots, \pm, 0, \cdots, 0)$ on the open unit ball in tangent space to the sphere at the pole.

Show that the overlap map $\varphi_j{}^{\pm} \circ (\varphi_i{}^{\pm})^{-1}$ is given by

$$\varphi_j{}^{\pm} \circ (\varphi_i{}^{\pm})^{-1}(y^1, \cdots, y^n) = \tag{2}$$
$$(y^1, \cdots, y^{i-1}, \pm\sqrt{1-(y^1)^2 - \cdots - (y^{i-1})^2 - (y^{i+1})^2 - \cdots - (y^n)^2}, y^{i+1}, \cdots, y^n)$$

Show that this atlas defines also a differentiable structure on S^n.

(c) Prove that the two differentiable structure defined in (a) and (b) above are isomorphic (they are equivalent). For the proof use the definition of diffeomorphism between smooth manifolds in Chern et all page 8.

2. (A) *Product Manifolds*: See Chern et al page 8 and page 6.
(B) *Division Algebras*: An algebra with a unit element is called a division algebra A. If for each non-zero element $a \in \mathcal{A}$ there exists its inverse $a^{-1} \in \mathcal{A}$, i.e., $a \cdot a^{-1} = a^{-1} \cdot a = 1$. The only possible associative division algebras are the real, complex and quaternion numbers. If associativity is dropped , octonions should be added to the above algebras. Let N denote the norm of $\mathcal{A}$ with $N(ab) = N(a)N(b), \quad N(a) > 0$ for all $a, b \in \mathcal{A}$. If $N(a) = 0$ then $a = 0$. Topologically unit norm real numbers are $S^0 \equiv \mathbf{Z_2}$, unit norm complex numbers are $S^{-1} \equiv U(1)$, unit norm quaternions are $S^3 \equiv SU(2)$, unit norm octonions are S^7. S^7 is not a group manifold because octonions are not associative.
(C) *Quaternions*: The quaternion $\mathbf{H}$ algebra is a four dimensional algebra over $\mathbf{R}$ with bases $\{e_0 = 1, e_1, e_2, e_3\}$ satisfying $e_i \cdot e_j = -\delta_{ij} + \epsilon_{ijk} e_k$. Here we assume that there is a summation over the repeated indexes from 0 to 3. A quaternion u can be written as $u = u^i e_i$ and its conjugate $\bar{u}$ is defined by $\bar{u} = u^0 - u^1 e_1 - u^2 e_2 - u^3 e_3$. The norm $N(u)$ of u is defined by $N(u)^2 = u \cdot \bar{u} = (u^0)^2 + (u^1)^3 + (u^2)^2 + (u^3)^2$. Hence the inverse of u is given by $u^{-1} = \frac{\bar{u}}{N(u)}$. A realization of of quaternions is given by the Pauli matrices. The basis e_i is expressed as $\{e_0 = 1, \ e_k == -i\, \sigma_k\}$ where σ_k's are the Pauli sigma matrices. Then a quaternion u in $\mathbf{H}$ is given by

$$u = u^0 - i \sum_{k=1}^{3} u^k \, \sigma_k = \begin{pmatrix} u^0 - iu^3 & -u^2 - iu^1 \\ u^2 - iu^1 & u^0 + iu^3 \end{pmatrix}$$

(D) *Milnor's exotic sphere* Σ^7: See Chern et al page 6, Example 4. Take mapping τ defined in this example as

$$\tau(u, v) = \left(\frac{u}{N(u)^2}, \frac{u^h v u^j}{N(u)} \right)$$

where $h = \frac{k+1}{2}$, $j = \frac{1-k}{2}$ and k is an odd integer. Prove that Σ_k^7 is differentiable (smooth) manifold. Here $k = \pm 1$ gives the standard differential structure (isomorphic) but all the others are not isomorphic to standard differential structures. Hence Σ^7 is homeomorphic but not diffeomorphic to S^7. See the original paper of Milnor , *Ann. Math.*, **64**, 399, (1956) (For differentiable structures on the sphere, see also J Milnor, *Am. J. Math.*, **81**, 962-972 (1959))

3. Prove that the real projective space RP^n is an analytic manifold. See Chern et al page 6 , Example 3.