MATH 443

PARTIAL DIFFERENTIAL EQUATIONS
Final Exam

January 12, 2020, Friday 09:00-11:00, SA-Z19

QUESTIONS: Solve any three of the following four problems. *In each problem explain what you are doing. I don't want you to use any formula to find the solutions. If you use a formula you have to prove it..*

[35]**1.** An initial value problem is given as follows:

$$
\begin{aligned}
&u_{tt} - c^2\, u_{xx} = 0, \quad t > 0, \quad x \in \mathbb{R}, \\
&u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \in \mathbb{R},
\end{aligned}
$$

where f and g are given functions and c is the speed of the wave. Is this problem well-posed? (existence, uniqueness and stability analysis).

[35]**2.** Solve the following initial value problem:

$$
\begin{aligned}
&(x+y)\, z_x + (x-y)\, z_y = 2xz, \\
&z(x,0) = e^{x^2+x},
\end{aligned}
$$

[35]**3.** Find the general solution the following partial differential equation

$$(D + 2D' - 3)(D - 2D' + 1)(D + 2D')^2\, z = 5\, e^x$$

where $D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial y}$

[35]**4.** Let

$$
\begin{aligned}
&\nabla^2 u \equiv u_{xx} + u_{yy} = 0, \quad (x,y) \in R, \\
&u|_{\partial R} = f,
\end{aligned}
$$

where R is a region in $\mathbb{R}^2$ and ∂R is the boundary of R which is a simple curve (no self intersection and has tangent vector at all its points). Prove that the function u takes it's maximum and minimum vales only on the boundary of R.

Solutions of the Final Exam problems: Jan. 12, 2020

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problem 1. An initial value problem is given as follows:

$$u_{tt} - c^2 u_{xx} = 0, \qquad t > 0,\ x \in \mathbb{R}$$

$$\left.\begin{aligned} u(t,0) &= f(x) \\ u_t(t,0) &= g(x) \end{aligned}\right\} x \in \mathbb{R}.$$

where f and g are given functions and c is the speed of the wave. Is this problem well-posed? (Existence, uniqueness and stability analysis)

This problem is solved in Class and also given in lecture notes on Wave equation (pages 68 – 70) problem 2. Solve the following initial value problem:

$$(x+y)z_x + (x-y)z_y = 2xz$$

$$z(x,0) = e^{x^2+x}$$

solution: Using the Lagrange Method we get

$$\frac{dx}{x+y} = \frac{dy}{x-y} = \frac{dz}{2xz}.$$

i) $$\frac{dx+dy}{2x} = \frac{dz}{2xz} \Rightarrow dx + dy = \frac{dz}{z}$$

$$\ln(z/z_0) = x+y \Rightarrow z = z_0 e^{x+y}$$

or $u = z\,e^{-x+y} = c_1$

ii) $$\frac{dx}{x+y} = \frac{dy}{x-y} \Rightarrow (x-y)dx - (x+y)dy = 0$$

$$v = \frac{x^2}{2} - \frac{y^2}{2} - 2xy = c_2.$$

Then the solution of the PDE is given as follows: There exist a function F such that

$$F(u,v) = 0$$

or $\quad u = f(v) \Rightarrow$

$$z = e^{x+y} f\left(\frac{x^2}{2} - \frac{y^2}{2} - 2xy\right) = e^{x+y} f(x^2 - y^2 - 2xy)$$

Initial condition

$$z(x,0) = e^{x} f\left(\frac{x^2}{2}\right) = e^{x^2+x}$$

$$\Rightarrow \quad f(x) = e^{x}$$

Hence

$$z = e^{x+y}\, e^{x^2-y^2-2xy} = e^{x^2-y^2-2xy+x+y}$$

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problem 3 $\quad$ Solve ~~the following~~ general solution of the following partial differential equation.

$$(D - 2D' - 3)(D - 2D' + 1)(D + 2D')^2 z = 5e^{x}$$

where $\quad D = \frac{\partial}{\partial x}, \quad D' = \frac{\partial}{\partial y}$

<u>Solution:</u> $z = z_h + z_p$, where z_h is the solution of the homogeneous equation and z_p is any particular solution of the pde.

$z_h = z_1 + z_2 + z_3$. where

$$(D + 2D' - 3) z_1 = 0$$

$$(D - 2D' + 1) z_2 = 0$$

$$(D + 2D')^2 z_3 = 0$$

Using the Lagrange Method we solve z_1 and z_2 as

$$z_1 = e^{3x} \phi_1 (2x - y)$$

$$z_2 = e^{-x} \phi_2 (2x + y)$$

where ϕ_1 and ϕ_2 are arbitrary functions. We can solve the function z_3 by using the Lagrange Method twice and get

$$z_3 = x \phi_3 (2x - y) + \phi_4 (2x - y)$$

where ϕ_3 and ϕ_4 are arbitrary function

We find a particular solution z_p by assuming it as a function of x alone, then

$$(D-3)(D+1)(D^2)\, z_p(x) = 5e^{x}$$

and let $z_p = Ae^{x} \Rightarrow$

$$(D^2-2D-3)D^2 z_p = (D^4-2D^3-3D^2)z_p$$

$$= -4Ae^{x} = 5e^{x} \Rightarrow \quad A = -5/4.$$

Therefore the general solution of the given pde is

$$z(x,y) = e^{3x}\phi_1(2x-y) + e^{-x}\phi_2(2x+y)$$
$$+ x\phi_3(2x-y) + \phi_4(2x-y) - \frac{5}{4}e^{x}$$

o

problem 4: Let $\nabla^2 u \equiv u_{xx} + u_{yy} = 0$, $(x,y) \in R$ and $u\big|_{\partial R} = f$, where R is a region in $\mathbb{R}^2$ and ∂R is the boundary of R which is a simple curve (no self ~~interaction~~ intersection and has tangent vector all its points). Prove that the function u takes it's max and min value on on the boundary of R:

This problem was solved in class and given also in the lecture notes on ~~Wave~~ Heat equation, pages 44 – 45.