MATH443
PARTIAL DIFFERENTIAL EQUATIONS
SECOND MIDTERM EXAM
SOLUTIONS
December 11, 2019 Wednesday 15:40-17:30, SA-Z02

QUESTIONS: Solve any three of the following four problems. *In each problem explain what you are doing. I don't want you to use any formula to find the solutions. If you use a formula you have to prove it..*

[35]**1**. Solve the equation

$$z_{xxx} - 2\, z_{xxy} - z_{xyy} + 2\, z_{yyy} = 2\, \sin y.$$

[35]**2**. Solve the equation

$$(x-y)(x^2\, z_{xx} - 2xyz_{xy} + y^2 z_{yy}) = 2xy(p-q)$$

[35]**3**. Solve the Cauchy problem

$$z_{xx} - z_{yy} = 0,$$

and $z(t, 2t) = e^t$, $z_x(t, 2t) = 2e^{2t}$

[35]**4**. Solve the following initial and boundary value problem and justify your solution.

$$\begin{aligned} &u_t = k u_{xx}, \quad t > 0, \; x \in (0, L), \; k > 0 \\ &u(0,t) = 0, \quad t \geq 0, \\ &u_x(L,t) = 0, \quad t \geq 0 \\ &u(x,0) = f(x). \end{aligned}$$

Problem 1

$$z_{xxx} - 2 z_{xxy} - z_{xyy} + 2 z_{yyy} = 2\sin y$$

$$(D^3 - 2D^2D' - DD'^2 + 2D'^3)\, z = 2\sin y$$

$$[D^2(D - 2D') - D'^2(D - 2D')]\, z = 2\sin y$$

$$(D^2 - D'^2)(D - 2D')\, z = 2\sin y$$

$$(D - D')(D + D')(D - 2D')\, z = 2\sin y$$

$$z = z_1 + z_2 + z_3 + h(y)$$

where

$$(D - D')z_1 = 0 \quad \Rightarrow \quad z_1 = f_1(x+y)$$

$$(D + D')z_2 = 0 \quad \Rightarrow \quad z_2 = f_2(x-y)$$

$$(D - 2D')z_3 = 0 \quad \Rightarrow \quad z_3 = f_3(2x+y)$$

and

$$h_{yyy} = 2\sin y \quad \Rightarrow \quad h(y) = \cos y.$$

$$\Rightarrow \quad z(x,y) = f_1(x+y) + f_2(x-y) + f_3(2x+y) + \cos y.$$

where f_1, f_2, and f_3 are arbitrary functions.

Problem 2

$$x^2 z_{xx} - 2xy\, z_{xy} + y^2 z_{yy} = 2xy\, \frac{p-q}{x-y}$$

a) Type of the PDE : $R = x^2,\ S = -2xy,\ T = y^2$

$$4RT - S^2 = 4x^2y^2 - 4x^2y^2 = 0 \qquad (\text{parabolic type})$$

b) Transformation to ξ and η.

$$R\lambda^2 + S\lambda + T = 0 \Rightarrow x^2\lambda^2 - 2xy\lambda + y^2 = 0$$

$$\lambda = y/x \Rightarrow \frac{\xi_x}{\xi_y} = \frac{y}{x} \Rightarrow x\xi_x - y\xi_y = 0$$

$$\frac{dx}{x} = -\frac{dy}{y} \Rightarrow \xi = xy \qquad \text{choose } \eta = x$$

$$\text{Jacobian} \quad J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} y & x \\ 1 & 0 \end{vmatrix} = -x$$

$J \neq 0$ if $x \neq 0$, let $z(x(\xi,\eta), y(\xi,\eta)) = \zeta(\xi,\eta)$

c) Transformation of the Equation

$$z_x = y\zeta_\xi + \zeta_\eta$$

$$z_{xx} = y^2\zeta_{\xi\xi} + 2y\zeta_{\xi\eta} + \zeta_{\eta\eta}$$

$$z_y = x\zeta_\xi, \quad z_{yy} = x^2\zeta_{\xi\xi}$$

$$z_{xy} = xy\zeta_{\xi\xi} + x\zeta_{\xi\eta} + \zeta_\xi$$

Then the DE changes to

$$\eta\zeta_{\eta\eta} = \frac{2\xi\zeta_\eta}{\eta^2 - \xi}$$

let $\eta = e^\rho$ then the above eqn reduces to

$$\frac{z_{\rho\rho}}{z_\rho} = 1 + \frac{2\xi}{e^{2\rho} - \xi} = 1 + \frac{2\xi e^{-2\rho}}{1 - \xi e^{-2\rho}}$$

$$\Rightarrow z_\rho = \alpha\left(e^\rho - \xi e^{-\rho}\right), \qquad \alpha = \text{const wrt } \rho. \text{ It may depend on } \xi$$

$$z_1(\xi,\eta) = \alpha(\xi)(\eta + \xi/\eta) + \beta(\xi)$$

or

$$z(x,y) = \alpha(xy)(x+y) + \beta(xy)$$

Here α and β are arbitrary functions

Problem 3

$$z_{xx} - z_{yy} = 0$$

$$z(t, 2t) = e^t$$

$$z_x(t, 2t) = 2e^{2t}$$

a) solution of the PDE.

$$(D^2 - D'^2)z = 0 \Rightarrow (D+D')(D-D')z = 0$$

$$z(x,y) = f(x+y) + g(x-y)$$

b) Initial conditions: First we discuss whether the initial curve γ: $x = t$, $y = 2t$ is a characteristic curve of the equation.

- The following level curves are the characteristic curves of the PDE

$$x - y = \text{constant}$$

$$x + y = \text{constant}$$

- $x = t$, $y = 2t$ curve does not satisfy any one of these equations. Hence γ is not a characteristic curve of the DE. Therefore there must be a unique solution of this Cauchy problem

$$z(t,2t) = f(3t) + g(-t) = e^{t} \qquad (1)$$

$$z_x(x,y) = f'(x+y) + g'(x-y)$$

$$z_x(t,2t) = f'(3t) + g'(-t) = 2e^{2t}$$

The last equation can also be written as

$$\frac{1}{3} f_t(3t) - g_t(-t) = 2e^{2t}$$

or

$$\frac{1}{3} f(3t) - g(-t) = e^{2t} + C \qquad (2)$$

where C is a constant. From (1) and (2) we find that

$$f(t) = \frac{3}{4}\left(e^{t/3} + e^{2t/3} + C\right)$$

$$g(t) = \frac{1}{4} e^{-t} - \frac{3}{4}\left(e^{-2t} + C\right)$$

$\Rightarrow$

$$z(x,y) = \frac{3}{4}\left(e^{\frac{x+y}{3}} - e^{-2(x-y)}\right) + \frac{3}{4} e^{\frac{2}{3}(x+y)} + \frac{1}{4} e^{-(x-y)}$$

problem 4

$$u_t = k u_{xx} \quad , \quad t > 0, \; x \in (0, L), \; k > 0$$

$$u(0,t) = 0 \quad , \quad t \geq 0$$

$$u_x(L,t) = 0 \quad , \quad t \geq 0$$

$$u(x,0) = f(x)$$

a) Formal solution: use the separation of variables

$$u(x,t) = \phi(x)\, G(t) \Rightarrow$$

$$\frac{1}{k}\frac{G_t}{G} = \frac{\phi_{xx}}{\phi} = -\lambda \qquad \lambda = \text{const}$$

$$\Rightarrow \quad G_t = -k\lambda G \quad \Rightarrow \quad G(t) = C e^{-k\lambda t}, \quad C = \text{const.}$$

$$\phi_{xx} + \lambda\phi = 0 \qquad 0 < x < L$$

From the boundary conditions, since they are homogeneous

$$\phi(0) = 0$$

$$\phi'(L) = 0$$

i) When $\lambda > 0$ $\quad \phi(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}\phi$

$$\phi(0) = A = 0$$

$$\phi_x(x) = +\sqrt{\lambda}\, B\cos(\sqrt{\lambda}x)$$

$$\phi_x(L) = \sqrt{\lambda}\, B\cos(\sqrt{\lambda}L) = 0$$

nontrivial case: $\quad \sqrt{\lambda}\, L = \frac{2n+1}{2}\pi$

$$\lambda = \left(\frac{2n+1}{2}\,\frac{\pi}{L}\right)^2 \qquad n = 0, 1, 2 \ldots$$

eigen function ⑥

$$\phi_n = \cos\left(\frac{2n+1}{2} \cdot \frac{\pi}{L} x\right) \qquad n = 0, 1, 2 \ldots$$

ii) when $\lambda = 0 \Rightarrow \phi = Ax + B$

$$\phi(0) = B = 0$$

$$\phi_x(L) = A = 0 \quad \text{trivial soln.}$$

There is only the trivial case

iii) when $\lambda < 0$. There is only the trivial case

$$\phi(x) = A e^{\sqrt{|\lambda|}\, x} + B e^{-\sqrt{|\lambda|}\, x}$$

$$\phi(0) = 0 \qquad A + B = 0$$

$$\phi_x(L) = \cancel{\sqrt{|\lambda|}\,(A - B) = 0}$$

$$= \sqrt{|\lambda|}\left(A e^{\sqrt{|\lambda|} L} - B e^{-\sqrt{|\lambda|} L}\right)$$

$$= \sqrt{|\lambda|}\, A \sinh(\sqrt{|\lambda|}\, L) = 0$$

$$\Rightarrow A = B = 0$$

$$u_n(x,t) = C_n e^{-k\lambda_n t} \cos(\sqrt{\lambda_n}\, x), \qquad \lambda_n = \left(\frac{2n+1}{2} \frac{\pi}{L}\right)^2$$

$n = 1, 2 \ldots$

superposition

$$u(x,t) = \sum_{n=0}^{\infty} C_n e^{-k\lambda_n t} \cos(\sqrt{\lambda_n}\, x)$$

Initial condition

$$u(x,0) = \sum_{n=0}^{\infty} c_n \cos(\sqrt{\lambda_n}\, x) = f(x)$$

the set of functions $\{\cos(\sqrt{\lambda_n}\, x)\}$ is an orthogonal set over $(0,L)$

$$\int_0^L \cos(\sqrt{\lambda_n}\, x)\cos(\sqrt{\lambda_m}\, x)\, dx = \begin{cases} 0 & m \neq n \\ L/2 \end{cases}$$

$$\Rightarrow \quad c_n = \frac{2}{L}\int_0^L f(x)\cos(\sqrt{\lambda_n}\, x)\, dx$$

Formal solution:

$$u(x,t) = \sum_{n=0}^{\infty} c_n e^{-k\lambda_n t} \cos(\sqrt{\lambda_n t}\, x) \qquad (1)$$

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} c_n \cos(\sqrt{\lambda_n}\, x) \qquad (2)$$

where $c_n = \frac{2}{L}\int_0^L f(x)\cos(\sqrt{\lambda_n}\, x)\, dx$, $n = 0,1,2,\ldots$

b) The first series is uniformly convergent if $f(x)$ is bounded in $(0,L)$. (proved in class)

The second series is uniformly convergent if f has continuous second derivatives in $(0,L)$ (done in the class)