SOLUTIONS

MATH443
PARTIAL DIFFERENTIAL EQUATIONS
FIRST MIDTERM EXAM

November 6, 2019 Wednesday 15:40-17:30, SA-Z02

QUESTIONS: Solve any three of the following four problems. *In each problem explain what you are doing.*

[35]1. (a) Find the general solution of the equation

$$(x-y)p+(y-x-z)q=z$$

(b) and the particular solution through the circle $z=1, \quad x^2+y^2=1$.

[35]2. Find all functions $\psi(x,y,z)$ so that the pfaffian differential equation $6x^2\,y\,z dx+2z\,x^3\,dy+\psi\,dz=0$ is integrable.

[35]3. (a) Verify that the Pfaffian differential equation

$$(y^2+yz)dx+(xz+z^2)dy+(y^2-xy)dz=0$$

is integrable. (b) Find the primitive of the differential equation.

[35]4. (a) Find a complete integral of the equation $p^2+4pq=z^2$. (b) show that this equation is compatible with the equation $q=a^2\,p$ and solve it. Here a is a nonzero real constant,

1 This is a homework 2 problem.

A) General solution: Using the Lagrange method

$$\frac{dx}{x-y} = \frac{dy}{y-x-z} = \frac{dz}{z}$$

i) $$\frac{dx+dy}{x-y+y-x-z} = \frac{dz}{z} \Rightarrow x+y+z = C_1$$

ii) $$\frac{dx-dy}{2(x-y)+z} = \frac{dz}{z}$$ let $u = x-y$

$$\Rightarrow \frac{du}{2u+z} = \frac{dz}{z} \Rightarrow \quad z\,du - (2u+z)dz = 0$$

$$z\,du - 2u\,dz - z\,dz = 0$$

$$z^3\, d\left(\frac{u}{z^2}\right) - z\,dz = 0 \Rightarrow d\left(\frac{u}{z^2}\right) - \frac{dz}{z^2} = 0$$

$$\Rightarrow \frac{x-y}{z^2} + \frac{1}{z} = C_2$$

Hence the general solution is

$$F\left(x+y+z, \frac{x-y}{z^2} + \frac{1}{z}\right) = 0$$

where F is any function

B) $z=1$; $x^2+y^2=1$. Inserting $z=1$ in the expression for C_1 and C_2 we get

$$\left.\begin{aligned} x+y &= C_1 - 1 \\ x-y &= C_2 - 1 \end{aligned}\right\} \quad \begin{aligned} x &= \tfrac{1}{2}(C_1 + C_2 - 2) \\ y &= \tfrac{1}{2}(C_1 - C_2) \end{aligned}$$

Using x and y in the circle equation

$$x^2+y^2=\frac{1}{4}(c_1+c_2-2)^2+\frac{1}{4}(c_1-c_2)^2=4$$

$$\Rightarrow \quad (c_1-1)^2+(c_2-1)^2=2.$$

or

$$(x+y+z-1)^2+\left(\frac{x-y+z}{zz}-1\right)^2=2.$$

This is the particular solution passing through the curve $z=1$, $x^2+y^2=1$.

2) Pfaffian differential equation

$\cancel{(y^2+yz)dx+(xz+z^2)dy}$

$$6x^2yz\,dx+2zx^3\,dy+\psi\,dz=0$$

is integrable if $\vec{X}\cdot \mathrm{Curl}(\vec{X})=0$

where $\vec{X}=6x^2yz\,\hat{i}+2zx^3\,\hat{j}+\psi\,\hat{k}$

Here ψ is a differentiable function of x, y, z.

$$\mathrm{Curl}\,\bar{X}=\begin{vmatrix}\hat{i} & \hat{j} & k\\ \partial_x & \partial_y & \partial_z\\ 6x^2yz & 2zx^3 & \psi\end{vmatrix}=(\psi_y-2x^3)\hat{i}-(\psi_x-6x^2y)\hat{j}$$

$$\bar{X}\cdot \mathrm{Curl}(\bar{X})=6x^2yz(\psi_y-\cancel{2x^3})-2zx^3(\psi_x-\cancel{6x^2y})$$

$$=6x^2yz\,\psi_y-2zx^3\,\psi_x$$

$$=2zx^2(3y\,\psi_y-x\,\psi_x)=0$$

or $\quad 3y\psi_y - x\psi_x = 0$

In order that the Pfaffian differential equation be integrable the function ψ must satisfy the above linear first order partial differential equation. Solution of this pde is:

$$\frac{dx}{-x} = \frac{dy}{3y} = \frac{d\psi}{0}$$

i) $\psi = c_1$ (const)

i) $3\frac{dx}{x} + \frac{dy}{y} = 0 \Rightarrow 3\ln x + \ln y = \text{const}$

or $\quad yx^3 = c_2$

The general solution is $\quad F(\psi, yx^3) = 0$

or $\quad \psi = f(yx^3, \cancel{z})$ where f is an arbitrary funct. Then the Pfaffian differential equation becomes

$$2z(3x^2y\,dx + x^3\,dy) + \psi\,dz = 0$$

or $\quad d(x^3y) + \psi\frac{dz}{2z} = 0$

let $\quad x^3y = \xi \Rightarrow \dfrac{2\,d\xi}{\psi(\xi,\cancel{z})} + \dfrac{dz}{z} = 0$

By using integrating factor we obtain $\Phi(\xi, z) = \text{const}$.

3 $(y^2+yz)\,dx + (xz+z^2)\,dy + (y^2-xy)\,dz = 0$

a) Checking the integrability.

$$\bar{X} = (y^2+yz)\hat{i} + (xz+z^2)\hat{j} + (y^2-xy)\vec{k}$$

$$\operatorname{Curl}(\bar{X}) = \begin{vmatrix} \hat{i} & \hat{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ y^2+yz & xz+z^2 & y^2-xy \end{vmatrix}$$

$$= (2y-2x-2z)\hat{i} + 2y\hat{j} - 2y\vec{k}$$

$$\bar{X}\cdot \operatorname{Curl}(\bar{x}) = (y^2+yz)(2y-2x-2z) + 2y(xz+z^2) - 2y(y^2-xy)$$

$$= \cancel{2y^3} - 2xy^2 - 2zy^2 + \cancel{2zy^2} - \cancel{2xyz} - \cancel{2yz^2} + \cancel{2yxz} + 2yz^2 - \cancel{2y^3} + 2y^2x = 0$$

Hence the Pfaffian differential equation is integrable

b) The Pfaffian differential equation is homogeneous therefore we let

$y = ux$, $z = vx$

Here u and v are new variables. Then the Pfaffian differential equation becomes

$$\frac{dx}{x} + \frac{(v+v^2)\,du + (u^2-u)\,dv}{u(u+v)(v+1)} = 0$$

$$\Rightarrow \frac{dx}{x} + \left[\frac{1}{u} - \frac{1}{u+v}\right] du + \left[\frac{1}{v+1} - \frac{1}{u+v}\right] dv = 0$$

$$\frac{dx}{x} + \frac{du}{u} + \frac{dv}{v+1} - \frac{du+dv}{u+v} = 0$$

$$\ln \frac{u(v+1)x}{u+v} = \text{const}$$

or $$\frac{u(v+1)x}{u+v} = c. \quad \text{(a constant)}$$

Inserting $u = y/x$, $v = z/x$ we get

$$\frac{y(z+x)}{z+y} = c.$$

The primitive of the differential equation is

$$\phi = \frac{y(z+x)}{z+y}$$

and the solution is

$$\phi = c.$$

4) a) $F(x, y, z, p, q) = 0$

where $F = p^2 + 4pq - z^2$.

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + qF_q} = \frac{dp}{-(F_x + pF_z)} = \frac{dq}{-(F_y + qF_z)}$$

$$\Rightarrow \quad \frac{dx}{2p+4q} = \frac{dy}{4p} = \frac{dz}{2z^2} = \frac{dp}{2zp} = \frac{dq}{2zq}$$

using the last two

$$\frac{dp}{2zp} = \frac{dq}{2zq} \Rightarrow \frac{dp}{p} = \frac{dq}{q} \Rightarrow q = ap$$

where a is any constant $\Rightarrow$

$$p^2(1+4a) = z^2 \Rightarrow p = \frac{\varepsilon z}{\sqrt{1+4a}}, \quad (4a+1 \neq 0) \quad \varepsilon = \pm 1$$

and $q = \dfrac{\varepsilon a z}{\sqrt{1+4a}} \Rightarrow$

$$dz = \frac{\varepsilon z}{\sqrt{1+4a}} dx + \frac{\varepsilon a z}{\sqrt{1+4a}} dy$$

or $$\frac{dz}{z} = \frac{\varepsilon}{\sqrt{1+4a}}(dx + a\,dy)$$

$$\ln z = \frac{\varepsilon}{\sqrt{1+4a}}(x + ay + b)$$

Here b is an arbitrary constant

So the complete integral of the nonlinear partial differential equation is

$$f(x,y,z,a,b) = -\ln z + \frac{\varepsilon}{\sqrt{1+4a}}(x+ay+b)$$

b) we have two equation

$$p^2 + 4pq = z^2$$
$$q = a^2 p$$

solve p and q we get

$$p = \frac{\varepsilon z}{\sqrt{1+4a^2}}, \quad q = \frac{\varepsilon a^2 z}{\sqrt{1+4a^2}}, \quad \varepsilon = \pm 1$$

$$\Rightarrow dz = \frac{\varepsilon z}{\sqrt{1+4a^2}}(dx + a^2 dy)$$

This Pfaffian differential equation in $\mathbb{R}^3$ is integrable. Hence the equation are compatible and the common solution is

$$\ln z = \frac{\varepsilon}{\sqrt{1+4a^2}}(x + a^2 y) + b$$

a, b are constant, $\varepsilon = \pm 1$.