MATH.443
PARTIAL DIFFERENTIAL EQUATIONS
MAKEUP EXAM 2

January 13, 2010 Wednesday 10.00-12.00

QUESTIONS WITH SOLUTIONS: Solve any four of the following five problems

[25]**1**. Reduce the equation

$$xu_{xx} - (x+y)u_{xy} + yu_{yy} = \frac{x+y}{x-y}(u_x - u_y)$$

to canonical form, and hence solve it.
SOLUTION: This PDE is of hyperbolic type. We find that $\xi = x + y$ and $\eta = xy$. Then the equation reduces to $u_{\xi\eta} = 0$. The general solution is

$$u(x,y) = f(x+y) + g(xy)$$

where f and g are arbitrary functions.

[25]**2**. **(a)**. Verify that the following equation is integrable and determine its primitive

$$(x^2 + xy + yz)dx - x(x+z)dy + x^2dz$$

SOLUTION: It is integrable and the primitive $\phi(x,y,z) = \ln(x+z) - \frac{y}{x} = C$.
(b). Find the integral curves of the equations:

$$\frac{dx}{x+y} = \frac{dy}{x+y} = \frac{dz}{-(x+y+2z)}$$

SOLUTION: First two gives $u(x,y) = x - y = c_1$. Adding the numerators and denominators of the firs two we get

$$\frac{dx+dy}{2(x+y)} = \frac{dz}{-(x+y+2z)}$$

Let $\xi = x + y$ then we get

$$\frac{d\xi}{2\xi} = \frac{dz}{-(\xi + 2z)}$$

This way we obtain

$$v(x, y) = \frac{1}{2}(x + y)^2 + 2z(x + y) = c_2$$

where c_1 and c_2 are constants and the integral curves are $u(x, y) = c_1$ and $v(x, y) = c_2$

[25]**3**. Find the solution the following initial and boundary value problem.

$$\begin{aligned} u_{tt} - c^2\, u_{xx} &= 0, \qquad (t > 0, 0 < x < \pi), \\ u(x, 0) &= 0, \qquad (0 \le x \le \pi), \\ u_t(x, 0) &= x(\pi - x), \qquad (0 \le x \le \pi), \\ u(0, t) &= u(\pi, t) = 0, \qquad (t \ge 0), \end{aligned}$$

Discuss the existence of the solution.

SOLUTION

$$u(x, t) = \sum_{n=1}^{\infty} a_n \sin nx \sin cnt$$

where

$$a_n = \frac{8}{c\,\pi n^4}, \quad n \text{ is odd}$$

We obtain a_n from the initial condition

$$x(\pi - x) = c \sum_{n=1}^{\infty} n\, a_n \sin nx$$

then

$$cna_n = \frac{2}{\pi} \int_0^{\pi} x(\pi - x) \sin nx dx = \frac{8}{\pi n^3}[1 - (-1)^n], \quad n = 1, 2, \cdots$$

with the above $a_n = \frac{8}{c\,\pi n^4}$ when $n =$ odd and $a_n = 0$ when $n =$ even then series corresponding to $u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}$ are all uniformly convergent. Hence the

solution given above satisfies both the wave equation and the initial condition. The the solution exists.

[25]**4. (a)**. Solve the equation

$$z_{xxxx} - 2\, z_{xxyy} + z_{yyyy} = 0$$

SOLUTION: This PDE can be written as

$$(D^2 - D'^2)^2\, z = 0$$

The the solution is

$$z(x,y) = f_1(x+y) + x\, f_2(x+y) + g_1(x-y) + x\, g_2(x-y)$$

where f_1, f_2, g_1 and g_2 are arbitrary functions.
(b). Solve the equation

$$z_{xy} = xy + 1$$

with $z(0,y) = 0$ and $z(x,0) = 0$
SOLUTION: Just by straight forward integration we obtain

$$z(x,y) = \frac{1}{4}\, x^2 y^2 + xy + f(x) + g(y)$$

where f and g are arbitrary functions. Using the conditions $z(0,y) = 0$ and $z(x,0) = 0$ we find $f = g = 0$. The the solution is

$$z_{x,y} = \frac{1}{4}\, x^2 y^2 + xy$$

[25]**5**. Find the formal solution of the the following initial and boundary value problem.

$$\begin{aligned}
u_t - k\,(u_{xx} + u_{yy}) &= 0, \qquad (t > o, 0 < x < a, 0 < y < b),\\
u(x,y,0) &= f(x,y), \qquad (0 \le x \le a, 0 \le y \le b),\\
u(0,y,t) &= u(a,y,t) = 0, \qquad (0 \le y \le b, t \ge 0),\\
u(x,0,t) &= u(x,b,t) = 0, \qquad (0 \le x \le a, t \ge 0)
\end{aligned}$$

where $k > 0$, and $f(x,y)$ is a given function.
SOLUTION: Done in the class.