MATH 543: METHODS OF APPLIED MATHEMATICS I

Final Exam: Solutions

04 January 2002 Friday 15.30-17.30SAZ:18

QUESTIONS:

(25)**1**. Show that

$$|\sin \alpha x| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2n\alpha x}{4n^2-1}$$

where α is any real number.

Solution: Since the function is even we shall use the even part of the orthonormal set $\{\sqrt{\frac{\alpha}{2\pi}}, \sqrt{\frac{\alpha}{\pi}} \cos n\alpha x\}$ for $n = 1, 2, \cdots$ over the interval $[-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}]$. Then

$$|\sin \alpha x| = a_0 + \sum_{k=1}^{\infty} a_k \cos k\alpha x$$

where

$$a_0 = \frac{\alpha}{\pi} \int_0^{\pi/\alpha} \sin \alpha x dx = \frac{2}{pi},$$

$$a_k = \frac{2\alpha}{\pi} \int_0^{\pi/\alpha} \sin \alpha x \cos k\alpha x dx = \frac{2}{\pi} \frac{1+(-1)^k}{1-k^2},$$

From the above expression we have $k = 2n, \quad n = 1, 2 \cdots$ and the result follows.

(25)**2**. Find a two term perturbation of

$$u' + u = \frac{1}{1+\varepsilon u}, \quad t > 0, \quad 0 < \varepsilon << 1$$

with $u(0) = 0$. (a) use regular perturbation (b) use the Poincare-Lindstedt method. Compare your results and discuss the validity of the approximations.

Solution: (a) Regular perturbation.

$$u(t) = u_0(t) + \varepsilon u_1(t) + O(\varepsilon^2)$$

where

$$u_0' + u_0 = 1, \quad u_0(0) = 0,$$
$$u_1' + u_1 + u_0\,(u_0' + u_0) = 0, \quad u_1(0) = 0,$$

Solving u_0 and u_1 we obtain that

$$u(t) = 1 - e^{-t} + \varepsilon(te^{-t} + e^{-t} - 1) + O(\varepsilon^2)$$

(b) Poincare'-Lindstaedt Method.

$$t = \frac{\tau}{w},$$
$$w = 1 + \varepsilon w_1 + O(\varepsilon^2),$$
$$u(\tau) = u_0 + \varepsilon u_1 + O(\varepsilon^2)$$

with the following perturbation equations

$$u_{0,\tau} + u_0 = 1, \quad u_0(0) = 0, \quad u_0(\tau) = 1 - e^{-\tau}$$
$$u_{1,\tau} + u_1 + 1 + w_1\, e^{-\tau} - e^{-\tau} = 0, \tag{1}$$

We choose $w_1 = 1$ and hence we get

$$u_1 = -1 + e^{-\tau}$$

hence

$$u(\tau) = 1 - e^{-tau} - \varepsilon(1 - e^{-\tau}) + O(\varepsilon^2)$$

and $\tau = (1+\varepsilon)\,t + O(\varepsilon^2)$ Both perturbation method gives uniformly valid approximation but the PL method gives more better approximation

(25)**3**. (a) Let

$$I(\lambda) = \int_a^b f(t)\, e^{\lambda\, g(t)}\, dt$$

where f is continuous and and g is smooth enough in the interval $[a, b]$. Furthermore the maximum of g occurs at $t = a$ (one of the end points of the integration interval) with $g'(t) < 0$ for all $t \in [a, b]$. Prove that the dominant term is

$$I(\lambda) \sim -\frac{f(a)\, e^{\lambda g(a)}}{\lambda g'(a)}$$

for $\lambda >> 1$

(b) Verify the following approximation for large λ

$$\int_1^2 \sqrt{3+t}\, e^{\frac{\lambda}{1+t}}\, dt \sim \frac{8}{\lambda}\, e^{\lambda/2}$$

Solution: The dominant term will at the point where the function g has maximium hence

$$I(\lambda) \sim f(a)\, e^{\lambda g(a)} \int_a^b e^{\lambda\, g'(a)(t-a)} dt \sim \frac{f(a) e^{\lambda g(a)}}{\lambda g'(a)} \int_0^{\lambda g'(b-a)} e^u\, du$$

since $\lambda >> 1$ the the integral above gives -1 hence the result follows.

(25)**4**. Solve the following boundary value problem:

$$u'' + w^2 u = f(x), \quad u(-\pi) = u'(\pi) = 0$$

where w is a positive constant.

Solution: Using the Green's function technique we find that

$$G(x,y) = \begin{cases} U_<, & x < y \\ U_>, & x > y \end{cases}$$

where

$$U_< = A\cos wx + B\sin wx, \quad U_> = C\cos wx + D\sin wx$$

Here A, B, C, D are constants (wrt x) to be determined. Using the boundary conditions we obtain

$$U_< = \frac{A}{\sin w\pi}\,\sin w(x+\pi), \quad U_> = \frac{D}{\sin w\pi}\,\cos w(x-\pi)$$

Here there is a condition $\sin w\pi \neq 0$ but this not so important. The result will not be effected by this constraint. Using the Continuity and jump conditions at $x = y$ we obtain

$$\begin{aligned} U_< &= -\frac{1}{w\cos 2w\pi}\,\cos w(y-\pi)\,\sin w(x+\pi), \\ U_> &= -\frac{1}{w\cos 2w\pi}\,\cos w(x-\pi)\,\sin w(y+\pi) \end{aligned}$$

then the solution is

$$\begin{aligned} u(x) = &-\frac{\cos w(x-\pi)}{w\cos 2w\pi}\int_{-\pi}^{x} f(y)\sin w(y+\pi)\,dy \\ &-\frac{\sin w(x+\pi)}{w\cos 2w\pi}\int_{x}^{\pi} f(y)\cos w(y-\pi)\,dy \end{aligned}$$

Here we must assume that $\cos 2w\pi \neq 0$. If $cos 2w\pi = 0$ it means that there exists nontrivial solution of the homogenous equation satisfying both boundary conditions (*Prove this statement*). In that case the above Green's function technique fails. We must modify the definition of the Green's function as we have done it in SET 5 page 8.