MATH 346
DIFFERENTIAL GEOMETRY II
Second Midterm Exam

April 21, 2008
Monday 10.40-12.30, SAZ-19

QUESTIONS: Solve any three of the following four questions

[35]**1**. Let $p \in S$. Then there exists a neighborhood $W \subset S$ of p such that if $\gamma : I \to W$ is a parametrized geodesic with $\gamma(0) = p$, $\gamma(t_1) = q$, $t_1 \in I$, and $\alpha : [0, t_1] \to S$ is a parametrized regular curve joining p to q, show that $\ell_\gamma \leq \ell_\alpha$, where ℓ_α denotes the length of the curve α. Furthermore, if $\ell_\gamma = \ell_\alpha$, then show that the trace of γ coincides with the trace of α between p and q.

[35]**2**. Let $p \in S$ and let $S_r(p)$ be a geodesic circle of center p and radius r. Let L be the arc length of $S_r(p)$ and A be the area of the region bounded by $S_r(p)$. Prove that

$$4\pi A - L^2 = \pi^2 r^4 K(p) + R$$

where $K(p)$ is the Gaussian curvature of S at p and

$$\lim_{r \to 0} \frac{R}{r^4} = 0$$

Thus , if $K(p) > 0$ (or < 0) and r small , $4\pi A - L^2 > 0$ (or < 0).

(35)**(3).** A regular parametrized curve $\alpha : [0, \ell] \to S$, where the parameter $s \in [0, \ell]$ is the arc length of α, is a geodesic if and only if , for every proper variation $h : [0, \ell] \times (-\varepsilon, \varepsilon) \to S$ of α, $L'(0) = 0$, where

$$L(t) = \int_0^\ell [< \frac{\partial h}{\partial s}, \frac{\partial h}{\partial s} >]^{1/2} \, ds$$

[35]**4.** (a) construct a Jacobi field along a geodesic of a unit sphere, (b) Define the conjugate point and find the locus of conjugate points in a unit sphere.