MATH 346
DIFFERENTIAL GEOMETRY II
Second Midterm Exam

April 28, 2010
Wednesday 10.40-12.30, SAZ-02

QUESTIONS: Choose any three from the following first four problems (1-4) and choose one from the last two problems, 5-6. *If you solve the bonus problem you can choose one from the first four and one from the last two problems*

[25]**1**. Fix a point $p_0 \in \mathbb{R}^2$ and define a family of maps $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$, $t \in [0, \ell]$, by $\varphi_t = t\, p_0 + (1-t)p$, $p \in \mathbb{R}^2$. Notice that $\varphi_0 = p$, $\varphi_1 = p_0$. Thus, φ_t is a continuous family of maps which starts with identity map and ends with a constant map p_0. Apply these considerations to prove that $\mathbb{R}^2$ is simply connected.

[25] **2**. Prove that a plane, regular, closed curve is convex if and only if it is simple and its curvature k does not change sign.

[25] **3**. Let $\alpha : [0, \ell] \to \mathbb{R}^2$ and $\beta : [0, \ell] \to \mathbb{R}^2$ be two differentiable plane curves, and let $p_0 \in \mathbb{R}^2$ be a point such that $p_0 \notin \alpha([0, \ell])$ and $p_0 \notin \beta([0, \ell])$. Assume that, for each $t \in [0, \ell]$, the points $\alpha(t)$ and $\beta(t)$ are closer than the points $\alpha(t)$ and p_0; i.e., $|\alpha(t) - \beta(t)| < |\alpha(t) - p_0|$. Prove that the winding number of α relative to p_0 is equal to the winding number of β relative to p_0

[25] **4**. Show that the map $\pi : \mathbb{R}^2 - \{0, 0\} \to \mathbb{R}^2 - \{0, 0\}$ given by

$$\pi(x.y) = (x^2 - y^2, 2xy), \quad (x, y) \in \mathbb{R}^2$$

is a two-sheeted covering map.

[25] **5**. The bending energy of a wire (a plane regular curve $\alpha : [0,1] \to \mathbb{R}^2$) is defined to be the integral of a squared curvature over the length of the curve, $\int_0^1 k^2(s)ds$. Recall that the curvature of a plane curve is also given by $\dot{\theta} = d\theta/ds$, where θ is the angle the unit tangent vector $\underline{\mathrm{T}}(s)$ makes with the $x-$ axis. What shape will a wire (curve) take to have minimum bending energy if the total turning (θ) of its tangent vector is fixed, 1/6, and the turning is zero at the end points? [Hint: Minimize the functional

$$J = \int_0^1 \frac{1}{2} (\dot{\theta})^2 \, ds$$

subject to

$$I = \int_0^1 \theta ds = \frac{1}{6}$$

with $\theta(0) = \theta(1) = 0$].

[25] **6**. Find the Killing vector fields of the right cylinder.

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[50] **Bonus Problem**. State and prove the Fary-Milnor theorem.

APPENDIX:

Let S be a regular surface and $X(u,v)$ be a parametrization of in the neighborhood of its point p where $(u,v) \in U \subset \mathbb{R}^2$. The normal vector N at P is orthogonal to X_u and X_v. Hence the Tangent space $T_p(S)$ at p is spanned by X_u and X_v. Then the components of the first fundamental form are given by

$$\begin{aligned} < X_u, X_u > &= E, \quad < X_u, N >= 0, \\ < X_u, X_v > &= F, \quad < X_v, N >= 0, \\ < X_v, X_v > &= G, \quad < N, N >= 1, \end{aligned}$$

The Gauss Equations:

$$\begin{aligned} X_{,uu} &= \Gamma^1_{11} X_{,u} + \Gamma^2_{11} X_{,v} + eN, \\ X_{,uv} &= \Gamma^1_{12} X_{,u} + \Gamma^2_{12} X_{,v} + fN, \\ X_{vv} &= \Gamma^1_{22} X_{,u} + \Gamma^2_{22} X_{,v} + gN, \end{aligned}$$

where

$$e =< X_{uu}, N >,\ f =< X_{uv}, N >,\ g =< X_{vv}, N >$$

are the coefficients of the **second fundamental form**, and

$$\begin{aligned} \Gamma^1_{11} &= \frac{G\,E_u - 2F\,F_u + F\,E_v}{2(EG - F^2)},\ \Gamma^2_{11} = \frac{2E\,F_u - E\,E_v - F\,E_u}{2(EG - F^2)}, \\ \Gamma^1_{12} &= \frac{E\,G_u - F\,E_v}{2(EG - F^2)}, \qquad\qquad \Gamma^2_{12} = \frac{E\,G_u - F\,E_v}{2(EG - F^2)}, \\ \Gamma^1_{22} &= \frac{2G\,F_v - G\,G_u - F\,G_v}{2(EG - F^2)},\ \Gamma^2_{22} = \frac{E\,G_v - 2F\,F_v + F\,G_u}{2(EG - F^2)}, \end{aligned}$$

are the **christoffel** symbols where $\Gamma^1_{12} = \Gamma^1_{21}$ and $\Gamma^2_{12} = \Gamma^2_{21}$

Weingarten equations:

$$\begin{aligned} N_{,u} &= \frac{fF - eG}{EG - F^2}\, X_{,u} + \frac{eF - fE}{EG - F^2}\, X_{,v}, \\ N_{,v} &= \frac{gF - fG}{EG - F^2}\, X_{,u} + \frac{fF - gE}{EG - F^2}\, X_{,v}, \end{aligned}$$

(1.a). Show that **Gauss' equations** can be written as

$$X_{,ij} = \Gamma^k_{ij}\, X_{,k} + h_{ij}\, N, \quad i,j = 1,2$$

where $X_1 = X_u$, $X_2 = X_u$, $X_{11} = X_{uu}$, $X_{12} = X_{uv}$, $X_{22} = X_{vv}$, $h_{11} = e$, $h_{12} = h_{21} = f$, $h_{22} = g$. Here repeated indices are summed from 1 to 2. The christoffel symbols are given by

$$\Gamma^i_{jk} = \frac{1}{2}\, g^{im}\, (g_{mj,k} + g_{mk,j} - g_{jk,m}) \tag{1}$$

where $g_{11} = E$, $g_{12} = g_{21} = F$, $g_{22} = G$ are the components of the first fundamental form or *components of the metric tensor* and h_{ij} are the components of the second fundamental form, g^{ij} are the components of the inverse of matrix g_{ik}, i.e., $g^{im}\, g_{mk} = \delta^i_k$.

(1.b). **Gauss and Mean Curvatures**

$$K = \frac{eg - f^2}{EG - F^2}, \quad H = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}$$

(1.c). Show that **Weingarten's equations** can be written as

$$N_{,i} = -g^{kl}\, h_{il}\, X_k, \quad i = 1,2. \tag{2}$$

Here the sum is both on l and on k.

(1.d). Show that **Codazzi's equations** equations can be written as

$$\frac{\partial}{\partial u_l}\, h_{ij} - \Gamma^k_{il}\, h_{kj} = \frac{\partial}{\partial u_j}\, h_{il} - \Gamma^k_{ij}\, h_{kl}, \quad i,j,l = 1,2 \tag{3}$$

or

$$\nabla_l\, h_{ij} = \nabla_j\, h_{il}, \quad i,j,l = 1,2 \tag{4}$$

where ∇ is covariant derivative. Here $u^i = u, u^2 = v$.

(1.e). **Covariant Derivative**: Let $\alpha : I = [a,b] \to S$ be curve in S with parameter $s \in I$, where $\alpha(0) = p \in S$ and $\alpha'(0) = v \in T_p(S)$. Let w be a

vector field along the curve α in S, so that $w = w^i\, X_{,i}$. Show that **Covariant derivative** of the vector field is given by

$$\frac{D\, w}{ds} = (\nabla_k\, w^i)\, \frac{du^k}{ds}\, X_{,i}$$

where

$$\nabla_k\, w^i = \frac{\partial w^i}{\partial u^k} + \Gamma^i_{kl}\, w^l, \quad k, i = 1, 2$$

(1.f). **Geodesics**: A curve $\alpha : I = [a, b] \to S$ in S with parameter $s \in I$ is a geodesic in S if it satisfies

$$\frac{D\, \alpha'}{ds} = 0,$$

Show that this is equivalent to

$$\frac{d^2 u^i}{ds^2} + \Gamma^i_{jk}\, \frac{du^j}{ds}\, \frac{du^k}{ds} = 0$$

(1.g).(**Serret-Frenet Equations**) Let $\alpha : [0, \ell] \to \mathbb{R}^3$ be a regular curve in $\mathbb{R}^3$, parametrized by its arc length parameter $s \in [0, \ell]$. Let $T(s)$, $n(s)$ and $b(s)$ be the unit tangent, normal and bi-normal vectors of α at the point $\alpha(s)$. They satisfy the Serret-Frenet equations

$$\begin{aligned} \dot{T} &= k(s)\, n, \\ \dot{n} &= -k(s)\, T - \tau(s)\, b, \\ \dot{b} &= \tau(s)\, n \end{aligned}$$

where $k(s)$ and $\tau(s)$ are the curvature and the torsion of α respectively.

(1.h). **Commutator of the covariant derivative**: Let $h : [0, \ell] \times (-\epsilon, \epsilon) \to S$ be a differential mapping and let $V(s, t)$ $((s, t) \in [0, \ell] \times (-\epsilon, \epsilon))$ be a vector field along h. Then

$$\frac{D}{\partial t}\, \frac{D}{\partial s}\, V - \frac{D}{\partial s}\, \frac{D}{\partial t}\, V = K(s, t)\, (\frac{\partial h}{\partial s} \times \frac{\partial h}{\partial t}) \times V$$

where $K(s, t)$ is the curvature of S at the point $h(s, t)$.