MATH 345
DIFFERENTIAL GEOMETRY I
Final Exam

January 13 , 2003
Friday 12.15-14.15, SAZ-04

QUESTIONS: Choose any four out of the following five questions

[25]**1**. (**a**) Let $F : M \to N$ be a mapping of surfaces M and N, Let $v, w \in T_p(M)$. Let ϕ be a $1-$ form and μ be a $2-$ form on N. Write down the definitions of the pullbacks $(F^* \phi)(v)$ and $(F^* \mu)(v, w)$. (**b**) Let ξ and η be $1-$ forms on N. Prove the following:

$$\begin{aligned} (1) \quad F^*(\eta + \xi) &= F^* \eta + F^* \xi, \\ (2) \quad F^*(\xi \wedge \eta) &= F^* \xi \wedge F^* \eta, \\ (3) \quad F^*(d \wedge \xi) &= d \wedge (F^* \xi) \end{aligned}$$

[25]**2**. (**a**) Prove that the ellipsoid defined by $M = \{(x, y, z) \in \mathbf{R}^3 | x^2 + y^2 + 4z^2 = 4\}$ is a regular surface in $\mathbf{R}^3$. (**b**) Find the unit normal vector field and the tangent plane of M at an arbitrary point $p = (x_0, y_0, z_0)$ of M (**c**) Find the normal line passing through the point p. and find the points of M where the normal line intersects. (**d**) Prove that the normal line intersects the $z-$ axis.

[25]**3**. Prove that the regular curves having the same curvature and torsion are congruent (They are the same curve up to an isometry)

[25]**4**. Let $X = x^2 U_1 + xy U_2$ and $Y = y U_1 - x U_2$ be two vector fields in $\mathbf{R}^2$. Here U_1 and U_2 are the elements of the standard basis of $\mathbf{R}^2$. Let $\nabla_Y X$ denote covariant derivative of X with respect Y. (**a**) Find $W = [X, Y] \equiv \nabla_X Y - \nabla_Y X$. (**b**) Find the vector field defined by

$$Z = [\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_W] X$$

[25]**5**. **(a)** Prove that a continuous function f on a compact region A in a surface M takes on a maximum value at some point of M **(b)** Prove that the paraboloid defined by $M = \{(x, y, z) \in \mathbf{R}^3 - \{0\} | z = x^2 + y^2\}$ is not a compact surface in $\mathbf{R}^3$.
(c) A critical point of a differentiable function : $M \to \mathbf{R}$ defined on a regular surface M is a point $p \in M$ such that $df_p = 0$. Let $f(p) = ||p - p_0||^2$, $p \in M$, $p_0 \in \mathbf{R}^3$ but not in M. Show that p is a critical point of f if and only if the line joining p to p_0 is normal to M.