MATH 343: PARTIAL DIFFERENTIAL EQUATIONS
Third Homework set
09 May 2002 Thursday 08.40

QUESTIONS:

Solutions of the boundary value problems by the method of Eigenfunctions

Let the boundary value problem be given as follows:

$$\nabla^2 \psi = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\, \psi = f(x,y) \quad in\, a\ region\, R \tag{1}$$

$$\psi = 0, \ on\, C = \ boundary\, of\, R, \tag{2}$$

where $f(x,y)$ is some real valued continuous function in R. We associate an eigenvalue problem to this boundary value problem as

$$\nabla^2 \varphi + \lambda\varphi = 0, in\, R \tag{3}$$

$$\varphi = 0, \ on\, C = \ boundary\, of\ R, \tag{4}$$

where λ is a parameter. An eigenvalue is a value of λ such that there exists a nontrivial function φ_λ with continuous second order derivatives which satisfy (3) and (4). The corresponding solution φ_λ is called an eigenfunction of the Laplace operator with boundary condition given in (4).

1. **(a)**. Prove that the Laplace operator is self adjoint. Hint: let u and v be two functions vanishing on C prove that

$$\int_R \int v\, \nabla^2\, u\, dA = \int_R \int u\, \nabla^2\, v\, dA$$

.

b. Prove that eigenfunctions belonging to different eigenvalues are ortghoganal, i.e.,

$$< \varphi_\lambda, \varphi_\mu >= \int_R\int \varphi_\lambda\, \varphi_\mu \, d\, A = 0, \quad \lambda \neq \mu$$

.

c. Let $h(x, y)$ be an arbitrary function in R, the series representation of this function is

$$h(x, y) = \Sigma_{n=1}\, a_n\, \varphi_n(x, y)$$

.

Prove that

$$a_n = \frac{< h, \varphi_n >}{< \varphi_n, \varphi_n >}$$

2. **a**. *Solution of the boundary value problem* : Prove that the solution of the boundarvalue problem stated in (1) and (2) in terms of the eigenfunctions of the Laplace operator is given by

$$\begin{aligned} \psi &= \Sigma_{n=1}\, c_n\, \varphi_n, & (5) \\ c_n &= -\frac{< f, \varphi_n >}{\lambda_n\, < \varphi_n, \varphi_n >} & (6) \end{aligned}$$

b. Show that

$$\psi(x, y) = \int_R\int G(x, y; \xi, \eta)\, f(\xi, \eta)\, d\xi\, d\eta$$

where

$$G(x, y; \xi, \eta) = -\Sigma_{n=1}\, \frac{\varphi_n(x, y)\, \varphi_n(\xi, \eta)}{\lambda_n\, < \varphi_n, \varphi_n >}$$

Interpret the meaning of the function G and show that **(i)**. it is symmetric with respect to the points (x, y) and (ξ, η). **(ii)**. G vanishes on C with respect to both variables. and **(iii)**. $\nabla^2_{(x,y)} G = \nabla^2_{(\xi,\eta)} G = 0$

3. Let the region R be the rectangle given by $R = \{(x, y) \in R^2 | 0 \leq x \leq a, 0 \leq y \leq b\}$.
(a). Prove that

$$\varphi_{mn}(x, y) = \sin(\frac{n\pi x}{a}) \sin(\frac{m\pi y}{b})$$

with $m, n = 0, 1, 2, \cdots$ are the eigenfunctions of the two dimensional boundary value problem in (3) and (4) with the eigenvalues

$$\lambda_{mn} = (\frac{n^2}{a^2} + \frac{m^2}{b^2})\pi^2$$

with $m, n = 1, 2, \cdots$.

b. Show that in part (a) above $< \varphi_{mn}, \varphi_{rs} >= 0$ if $(m, n) \neq (r, s)$ and $< \varphi_{mn}, \varphi_{mn} >= \frac{ab}{4}$.
c. Prove that

$$\psi(x, y) = -\Sigma_{m=1} \Sigma_{n=1} \frac{C_{mn}}{\lambda_{mn}} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

where

$$C_{mn} = \frac{4}{ab} \int_R \int f(x, y) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \, dx \, dy$$

with $m, n = 1, 2, \cdots$
d. Find the solution of the boundary value problem when $f = -2$ in R.

4. Discuss the solution of the boundary value problem stated in (1) and (2) when the boundary of R is a circle with radius a and $f = -2$ in R.