MATH. 343
PARTIAL DIFFERENTIAL EQUATIONS
FIRST MIDTERM EXAM

March 12, 2002 Tuesday 13:40-15.30 , SAZ-20

QUESTIONS:

1. Find the solutions of the following Pfaffian differential equations (a) $(y+z)dx+dy+dz=0$ and (b) $(y^2+z^2)dx+xydy+zxdz=0$.

solution: (a) $x+\ln(y+z)=c$ where c is a constant and $y+z>0$. (b) You may solve this Pfaffian equation by using the homogeneity of the coefficients or by inspection. It can written as

$$(y^2+z^2)dx+x(ydy+zdz)=0$$

or

$$\frac{dx}{x}+\frac{ydy+zdz}{x^2+y^2}=0$$

which can be integrated easily. It gives

$$x^2\,(y^2+z^2)=c$$

where c is a positive constant.

2. Find the general solution of the first order pde

$$(p-q)(x+y)=z.$$

Find the solution which passes through the curve C described by $C=\{(x,y,z)\in\mathbf{R}^3|x+y+z=0,z^2=x\}$.

solution: Using the Lagrange method we get

$$\frac{dx}{x+y}=-\frac{dy}{x+y}=\frac{dz}{z}$$

Solving these equation we get.

$$u = x + y = c_1, \qquad v = \ln z + \frac{y}{x+y} = c_2$$

Here c_1 and c_2 are arbitrary constants. The first part of the problem was to determine the general solution. It is $F(u,v) = 0$, where F is an arbitrary differentiable function of u and v. Or,

$$\ln z + \frac{y}{x+y} = f(x+y)$$

where $f(\xi)$ is an arbitrary differentiable function of its arguement. The second part of the problem is to determine this function if an initial curve is given. Let $x+y+z=0$ and $x = z^2$ be the initial curve then we get

$$\ln z + \frac{z+z^2}{z} = f(-z)$$

or

$$f(z) = \ln(-z) + 1 - z$$

hence the solution is

$$\ln z = 1 - \frac{y}{x+y} - x - y + \ln(-x-y)$$

Or

$$z = -(x+y)e^{-x-y}\, e^{\frac{x}{x+y}}$$

where $x+y<0$ and $z>0$.

3. Find the complete integrals of the equation $(p^2+q^2)y = qz$.

solution: We use the following set of equations

$$\frac{dx}{2py} = \frac{dy}{2qy - z} = \frac{dz}{qz} = \frac{dp}{pq} = \frac{dq}{-p^2}$$

they have solutions
(i) Using the last two we get $p^2+q^2 = a > 0$. Then we have $q = \frac{ay}{z}$ and $q = \sqrt{a - \frac{a^2y^2}{z^2}}$. Inserting these in $dz = pdx + qdy$ we get

$$y^2 + (x-b)^2 = \frac{z^2}{a}$$

(ii) Using the third and fourth one we find that $p = az$ where a is nonzero constant. Using this and the equation itself we find

$$q = \frac{z + \pm\sqrt{z^2 - 4a^2z^2y^2}}{y}$$

Hence using $dz = pdx + qdy$ we get

$$dz = adx + \frac{1 + \pm\sqrt{1 - 4a^2y^2}}{y}dy$$

Integration yields another complete integral.

4. Find the solution of the equation $p^2 + q^2 = 4z$ which reduces to $z = x^2 + 1$ when $y = 0$.

solution: Using the method of characteristics We have

$$\dot{x} = 2p, \;\; \dot{y} = 2q, \;\; , \dot{z} = 8z, \;\; \dot{p} = 4p, \;\; \dot{q} = 4q$$

We solve these with the initial conditions $x = v, y = 0, z = v^2 + 1$ the remaining two initial conditions are found from $z_{,v} = px_{,v} + qy_{,v}$ and the differential equation itself. They are $p = 2v$ and $q = \pm 2$. With these initial conditions the above equations have the solution

$$z = x^2 + (1 \pm y)^2$$