MATH. 343
PARTIAL DIFFERENTIAL EQUATIONS
FINAL EXAM

May 27 , 2002 Monday 15.30-17.30, SAZ-20

QUESTIONS: **SOLVE ONLY FOUR PROBLEMS**

[25]**1**. If u is a function of x, y, z which satisfies the partial differential equation

$$(y-z)u_x + (z-x)u_y + (x-y)u_z = 0,$$

Show that u contains x, y, and z only in combinations $x+y+z$ and $x^2+y^2+z^2$

solution: See Sneddon page 55 example 2

[25]**2**. Find a complete integral of the equation

$$z = x\, z_x + y\, z_y + F(z_x, z_y)$$

where F is any differentiable function of its arguements.

solution: See Sneddon page 72 *the Clairaut equation.*

[25]**3**. A function u satisfies the nonhomogenous wave equation

$$u_{xx} - u_{yy} = f(x, y),$$

and the initial conditions $u(x,y) = u_y(x,y) = 0$ when $y = 0$. Show that

$$u(x,y) = -\frac{1}{2}\int_T\int f(\xi,\eta) d\xi\, d\eta,$$

where T is the triangle cut out from the upper half of the $\xi\eta-$ plane by two characteristics through the point (x, y).

solution: See page 225 Problem 4. A straight forward application of the Riemann Green's function method done in the class in detail.

[25]**4**. Let $u(x,t)$ be a differentiable function satisfying the diffusion equation $u_t = k u_{xx}$ at all points $x \in I = (0, a)$ and $t > 0$ where k is a positive constant. The function u vanishes at end points of the interval I for all $tge0$

and satisfies the initial condition $u(x,0) = f(x)$ for all $x \in [0,a]$ where $f(x)$ is a continuous function on $[0,a]$.
(a) Find the formal solution
(b) Prove that the above formal solution satisfies the diffusion equation (everywhere in I and for $t > 0$) and find the minumum set of conditions on $f(x)$ so that u satisfies also the initial condition.

solution: See your Lecture notes.

[25]**5**. If $u_{xx} + u_{yy} = 0$ for $x > 0$ and $u(0,y) = f(y)$.
(a) Show by using the method of Fourier transform that

$$u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|\xi|x - i\xi y}\, d\xi$$

where $F(\xi)$ is the Fourier transform of $f(y)$.
(b) Prove that

$$u(x,y) = \frac{x}{\pi} \int_{-\infty}^{\infty} \frac{f(y')dy'}{x^2 + (y-y')^2}$$

solution: (a) Multiply the equation by $e^{i\xi y}$ and integrate with respect to y over $(-\infty, \infty)$ we get (using integration by parts two times and using the boundary condition $u = 0$ as $|y| \to \infty$)

$$\bar{u}_{xx} - \xi^2\, \bar{u} = 0$$

where

$$\bar{u}(x,\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi y}\, u(x,y)\, dy$$

we obtain

$$\bar{u}(x,\xi) = A(\xi)e^{-\xi x} + B(\xi)e^{\xi x}$$

To have a finite solution as $x \to \infty$, $(x > 0)$, we must let $B = 0$. Hence we have

$$\bar{u}(x,\xi) = A(\xi)\, e^{-\xi x}$$

Then by using the invres Fourier transform we get

$$u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(\xi)\, e^{-\xi x - i\xi y}\, d\xi, \qquad (1)$$

where

$$u(0,y) = f(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(\xi)\, e^{-\xi y}\, d\xi$$

or

$$A(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi y}\, f(y) dy \tag{2}$$

(Here $A(\xi) = F(\xi)$).
(b) Insert $A(\xi)$ in (2) in to the expression (1) we obtain the required result.