

MATH. 337
INTRODUCTION TO SOLITON THEORY

May 18 , 2009 Monday 14.40-16.30, SAZ-19

QUESTIONS:

[35]1. a) Let Λ be an algebra of Laurent series. Let $L \in \Lambda$ be given by

$$L = \sum_{i=-\infty}^{\infty} u_i(x, t) p^i$$

where u_i are differentiable functions of x and t . A bracket $\{, \} : \Lambda \times \Lambda \rightarrow \Lambda$ is defined as

$$\{f, g\} = p \left(\frac{\partial f}{\partial p} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} \right)$$

for all $f, g \in \Lambda$. Lax equation on Λ is given by

$$L_{,t} = \{A, L\}$$

where L is called the Lax operator and $A = (L^n)_{\geq 1}$. Here $()_{\geq 1}$ denotes that in the Laurent series of L consider only terms p^k with $k \geq 1$. Let $L = p + u(x, t) + v(x, t) p^{-1}$. Find the evolution equations satisfied by u and v when $n = 1, 2$.

[35]2. Find a functional $A = \iint L dx dt$ whose extremum is achieved by the solutions of the evolution equation $u_t - 10u^3 u_x + u_{xxx} = 0$. **Hint:** Find the Lagrange function L such that the Euler-Lagrange equation, $E(L) = 0$, gives this evolution equation.

[35]3. Given the sine-Gordon equation in the form $u_{xt} = \sin u$ verify the following conservations laws: $T_{i,t} = X_{i,x}$, $i = 1, 2, 3, 4 \dots$. With

$$\begin{aligned} X_1 &= \frac{1}{2} u_t^2, & T_1 &= 1 - \cos u, \\ X_2 &= 1 - \cos u, & T_2 &= \frac{1}{2} u_x^2, \\ X_3 &= -u_x^2 \cos u, & T_3 &= \frac{1}{4} u_x^4 - u_{xx}^2, \end{aligned}$$

$$\begin{aligned}
X_4 &= \frac{1}{4}u_t^4 - u_t^2, \quad T_4 = -u_t^2 \cos u, \\
X_5 &= -\left(\frac{1}{9}u_x^4 - \frac{4}{3}u_{xx}^2\right) \cos u, \quad T_5 = \frac{1}{6}u_x^6 - \frac{2}{3}u_x^2 u_{xx}^2 + \frac{8}{9}u_x^3 u_{xxx} + \frac{4}{3}u_{xxx}^3 \\
X_6 &= \frac{1}{6}u_t^6 - \frac{2}{3}u_t^2 u_{tt}^2 + \frac{8}{9}u_t^3 u_{ttt} + \frac{4}{3}u_{ttt}^3, \quad T_6 = -\left(\frac{1}{9}u_t^4 - \frac{4}{3}u_{tt}^2\right) \cos u
\end{aligned}$$

Hint: First prove the following: If $X(x, t), T(x, t)$ satisfy the conservation law $X_{,x}(x, t) = T_{,t}(x, t)$ then $T(t, x), X(t, x)$ satisfy the conservation law $T_{,x}(t, x) = X_{,t}(t, x)$.