MATH337 : INTRODUCTION TO SOLITON THEORY.

Spring 2012 : Lecture 5.

1. Hamiltonian Formulation of the KdV Equation

2. AKNS Formulation

3. Lax Formulation with Poisson Bracket

Hamiltonian formulation of the KdV eqn

a) Poisson Bracket: Poisson manifold.

Let $P, Q, \ldots$ be functional defined by

$$P = \int_{-\infty}^{\infty} \mathcal{P}\, dx$$

where $\mathcal{P} = \mathcal{P}(u, u_x, \ldots) \equiv \mathcal{P}[u]$. Here we assume $u, u_x, \ldots \to 0$ as $|x| \to \infty$ and all functionals $\mathcal{P} \to 0$ as $|x| \to \infty$. Let all the functionals define a space F. Define a function $\{\,,\,\}: F \times F \to F$ called Poisson bracket. F with $\{\,,\,\}$ called the Poisson manifold. Let $P, Q \in F$ then

$$\{P, Q\} = \iint_{-\infty}^{\infty} dx\, dy\, \frac{\delta \mathcal{P}}{\delta u(x)}\, \mathcal{D}_{xy}\, \frac{\delta \mathcal{Q}}{\delta u(y)}.$$

where $\frac{\delta \mathcal{P}}{\delta u} = E(\mathcal{P})$ where E is the variational derivative

$$E(\mathcal{P}) = \frac{\partial}{\partial u} - \frac{d}{dx}\frac{\partial}{\partial u_x} + \frac{d^2}{dx^2}\frac{\partial}{\partial u_{xx}} + \ldots$$

$$= \sum_{k=0} D_x^{i} \frac{\partial}{\partial u_{ki}} (-1)^{k} = \sum_{i=0} (-D_x)^{i} \frac{\partial \mathcal{P}}{\partial u_i}.$$

$\mathcal{D}$ is an operation (called the Hamilton operator)

$$\mathcal{D}_{xy} = \mathcal{D}_y\, \delta(x-y)$$

$$\{P,Q\} = \int_{-\infty}^{\infty} dx \, \frac{\delta P}{\delta u(x)} \, D_x \, \frac{\delta Q}{\partial u(x)}$$

here $\delta(x)$ is the Dirac-delta distribution.

o) $\delta(x-y) = 0 \quad x \neq y$
$\quad = \infty, \; x = y$

i) $\int_{-\infty}^{\infty} dy \, \delta(x-y) = 1$

ii) $\int_{-\infty}^{\infty} dy \, f(y) \delta(x-y) = f(x)$

iii) $\int_{-\infty}^{\infty} dy \, f(y) \, \partial_y \delta(x-y) = -\partial_x f(x)$

Poisson bracket satisfies ($D_x = \partial_x$ for KdV)

a) $\{P,Q\} = -\{Q,P\}$ skew symmetric

b) $\{P,\{Q,R\}\} + \dots = 0$ Jacobi identity

proofs:

a) integration by parts.

b) $\{P,Q\} = -\int_{-\infty} dx \left(\partial_x \frac{\delta P}{\delta u(x)}\right) \frac{\delta Q}{\delta u(x)}$

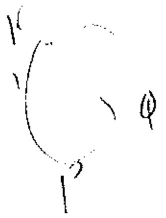
$$\{P,\{Q,R\}\} = -\int_{-\infty}^{\infty} dx \; \partial_x \frac{\delta P}{\delta u(x)} \; \frac{\delta}{\delta u} \{Q,R\}$$

$$= \iint_{-\infty} dx\, dy \; \partial_x \frac{\delta P}{\delta u(x)} \cdot \frac{\delta}{\delta u(x)} \left[\left(\partial_y \frac{\delta Q}{\delta u(y)}\right) \frac{\delta R}{\delta u(y)}\right]$$

$$= \iint dx\, dy \; \partial_x \frac{\delta P}{\partial u(x)} \left[\partial_y \frac{\delta^2 Q}{\partial u(x) \delta u(y)} \frac{\delta R}{\delta u(y)} + \partial_y \frac{\delta Q}{\delta u(y)} \frac{\delta^2 R}{\delta u(x) \delta u(y)}\right.$$

$$\{P, \{\Phi, R\}\} = \iint dx\,dy\; \partial_x \frac{\delta P}{\delta u(x)} \left[\partial_y \frac{\delta \Phi}{\delta u(y)} \frac{\delta^2 R}{\delta u(x)\,\delta u(y)} - \partial_y \frac{\delta R}{\delta u(y)} \frac{\delta^2 \Phi}{\partial u(y)\,\delta u(y)} \right]$$

$$\{P, \{\Phi, R\}\} + \{\Phi, \{R, P\}\} + \{R, \{P, \Phi\}\} = 0$$



$$\begin{aligned} = \iint dx\,dy \Big\{ & \partial_x \frac{\delta P}{\delta u(x)}\, \partial_y \frac{\delta \Phi}{\delta u(y)}\, \frac{\delta^2 R}{\delta u(x)\,\delta u(y)} \\ & - \partial_x \frac{\delta P}{\delta u(x)}\, \partial_y \frac{\delta R}{\delta u(y)}\, \frac{\delta^2 \Phi}{\delta u(y)\,\delta u(x)} \\ & + \partial_x \frac{\delta \Phi}{\delta u(x)}\, \partial_y \frac{\delta R}{\delta u(y)}\, \frac{\delta^2 P}{\delta u(x)\,\delta u(y)} \\ & - \partial_x \frac{\delta \Phi}{\delta u(x)}\, \partial_y \frac{\delta P}{\delta u(y)} \cdot \frac{\delta^2 R}{\delta u(y)\,\delta u(x)} \\ & + \partial_x \frac{\delta R}{\delta u(x)}\, \partial_y \frac{\delta P}{\delta u(y)} \cdot \frac{\delta^2 \Phi}{\delta u(x)\,\delta u(y)} \\ & - \partial_x \frac{\delta R}{\delta u(x)}\, \partial_y \frac{\delta \Phi}{\delta u(y)} \cdot \frac{\delta^2 P}{\delta u(x)\,\delta u(y)} \Big\} = 0 \end{aligned}$$

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1)
$$\int dy \{u(y), \varphi\} = \int dy \int \frac{\delta u}{\delta u(z)} \partial_z \frac{\delta \varphi}{\partial u(z)} dz$$
$$= \int dy \int \delta(y-z) \partial_z \frac{\delta \varphi}{\partial u(z)} dz$$
$$= \int dy\, \partial_y \frac{\delta \varphi}{\partial u(y)}$$

$$\Rightarrow \quad \{u(y), \varphi\} = \partial_y \frac{\delta \varphi}{\partial u(y)}$$

2)
$$\int_{-\infty}^{\infty} \{u(x), u(y)\}\, dy = \iint \frac{\delta u(x)}{\delta u(z)} \partial_z \frac{\delta u(y)}{\delta u(z)}\, dz\, dy$$
$$= \iint \delta(x-z)\, \partial_z \delta(y-z)\, dz\, dy$$
$$= \int\int \partial_x \delta(y-x)\, dy$$

$$\Rightarrow \quad \{u(x), u(y)\} = \partial_x \delta(y-x)$$

$$H = \int \mathcal{H}$$

3) KdV
$$\frac{\partial u}{\partial t} = \mathcal{D} \frac{\delta H}{\delta u} = \mathcal{D}\, E(\mathcal{H})$$
$$= \partial_x(-u_{xx} + 3u^2)$$

$$u_t - 6uu_x + u_{xxx} = 0 \checkmark$$

KdV equation in Hamiltonian form.

$$\{P, Q\} = \int_{-\infty}^{\infty} dx \, \frac{\delta P}{\delta u(x)} \, \partial_x \frac{\delta Q}{\delta u(x)}$$

let $P = \int_{-\infty}^{\infty} u(y,t)\, dy \qquad \frac{\delta P}{\delta u(x)} = E(u) = 1$

$$\int dy \{ u(y), Q \} = \int_{-\infty}^{\infty} dx \, \frac{\delta P}{\delta u(x)} \, \partial_x \frac{\delta Q}{\delta u(x)} = \int_{-\infty}^{\infty} dx \, \partial_x \frac{\delta Q}{\delta u(x)}$$

$$= \cancel{\int_{-\infty}^{\infty} dx \, \delta(x-y) \, \partial_x \frac{\delta Q}{\delta u(x)}}$$

$$\cancel{\{ u(x), P \} = \partial_x \frac{\delta P}{\delta u(x)}}$$

$$\Rightarrow \quad \{ u(x), Q \} = \partial_x \frac{\delta Q}{\delta u(x)}$$

KdV. $\quad u_t = 6uu_x - u_{xxx} = \partial_x (3u^2 - u_{xx})$

$$= \partial_x \frac{\delta H}{\delta u(x)} = \partial_x \, E(\mathcal{H}_2)$$

$$H = \underbrace{\int_{-\infty}^{\infty} \left(u^3 + \frac{1}{2} u_x^2 \right) dx}_{\mathcal{H}_2}$$

$$E(\mathcal{H}_2) = 3u^2 - \partial_x u_x = 3u^2 - u_{xx} \quad \checkmark$$

let $\quad Q = \int_{-\infty}^{\infty} u(y)\, dy$

$$\int_{-\infty}^{\infty} \{ u(x), u(y) \}\, dy = \partial_x$$

$$\Rightarrow \{ u(x), u(y) \} = + \partial_x \delta(x-y)$$

Hence for the KdV equation we have

$H = \int_{-\infty}^{\infty} [\, u^3 + \frac{1}{2} u_x^2 \,]\, dx$ Hamiltonian functional

$\mathcal{D} = D_x$ Hamilton operator

$$\frac{\partial u}{\partial t} = \mathcal{D} \frac{\delta H}{\delta u} = \mathcal{D} E(\mathcal{H})$$

For a given DE one investigates the pair $(\mathcal{D}, H)$ so that

$$\frac{\partial u}{\partial t} = \mathcal{D} \frac{\delta H}{\delta u}$$

Hamiltonian formulation of the KdV equation
For KdV there is a second pair $(\mathcal{D}_2, H_2)$ so that

$$\frac{\partial u}{\partial t} = \mathcal{D}_2 \frac{\delta H_2}{\delta t} \,, \qquad \begin{aligned} &\mathcal{D}_2 = D_x^3 - 4u D_x - 2u_x \\ &H_2 = \int_{-\infty}^{\infty} u\, dx \,, \quad \mathcal{H}_2 = u \end{aligned}$$

this means that

$$\{u(x), u(y)\}_2 = \mathcal{D}_2\, \delta(x-y)$$

poisson bracket

$$\{P, \varphi\}_2 = \int_{-\infty}^{\infty} dx\, \frac{\delta P}{\delta u}\, \mathcal{D}_2\, \frac{\delta \varphi}{\delta u}$$

$$= - \int_{-\infty}^{\infty} dx\, \frac{\delta \varphi}{\delta u}\, \mathcal{D}_2\, \frac{\delta P}{\delta u}$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} f\, D_2\, g \; dx = - \int_{-\infty}^{\infty} g\, D_2 f \qquad \text{"skew adjoint"}$$

$$\int_{-\infty}^{\infty} f\, M\, g\; dx = \int_{-\infty}^{\infty} g\, M^{\#} f\; dx$$

$M^{\#}$ adjoint of M

for Hamilton operators $\mathcal{D}^{\#} = -\mathcal{D}$

skew adjoint.

Hence: $\mathcal{D}_2^{*} = -\mathcal{D}_2$

proof: $\int f\,(g_{xxx} - 4u g_x - 2u_x g)$

$$= \int (-f_x g_{xx} + 4 f_x u g + 4u_x f g - 2u_x f g)\, dx$$

$$= \int g\,[-f_{xxx} + 4 f_x u g + 2u_x f g]$$

$$= -\int g\,(D^3 - 4uD - 2u_x) f\; dx \Rightarrow \mathcal{D}_2^{*} = -\mathcal{D}$$

$$\{ P, \varphi \}_2 = \int \frac{\delta P}{\delta u} \mathcal{D}_2 \frac{\delta \varphi}{\delta u} \, dx$$

satisfies also the Jacobi identity.

<u>prove it</u>.

<u>Lemma</u>

$$\frac{dP}{dt} = \{P, H_1\}_2 = \{P, H_2\}_1.$$

proof:

$$\{P, H_2\}_1 = \int_{-\infty}^{\infty} dx \, \frac{\delta P}{\delta u} \, D_1 \frac{\delta H_2}{\delta u}$$

$$= \int_{-\infty}^{\infty} \frac{\delta P}{\delta u} \, u_t = \int_{-\infty}^{\infty} dx \, E(\mathcal{P}) \, u_t$$

$$= \int_{-\infty}^{\infty} \left[\frac{\partial \mathcal{P}}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{P}}{\partial u_x}\right) + \cdots \right] u_t$$

$$= \int_{-\infty}^{\infty} dx \left[\frac{\partial \mathcal{P}}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{P}}{\partial u_x}\right) u_t + \frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{P}}{\partial u_{xx}}\right) u_t + \cdots \right.$$

$$= \int_{-\infty}^{\infty} dx \left[\frac{\partial \mathcal{P}}{\partial u} + \frac{\partial \mathcal{P}}{\partial u_x} u_{tx} + \frac{\partial \mathcal{P}}{\partial u_{xx}} u_{txx} + \cdots \right]$$

$$= \int_{-\infty}^{\infty} dx \, \frac{d\mathcal{P}}{dt} = \frac{dP}{dt} \quad \checkmark$$

If P is a conserved quantity then $\frac{dP}{dt} = 0$

$\{P, H_2\}_1 = 0$ and $\{P, H_1\}_2 = 0$

Lemma: all H_n's are conserved quantities, Hence

$$\{H_n, H_2\}_1 = \{H_n, H_1\}_2 = 0$$

$n=3$

$$D_2 \frac{\delta H_2}{\delta u} = D_1 \frac{\delta H_3}{\delta u}$$

$$= \partial_x \left(u_{xxxx} + 10u^3 + 5u_x^2 - 10(uu_x)_x \right)_x$$

. . .

KdV Hierachy

$$\frac{du}{dt_n} = D_2 \frac{\delta H_{n-1}}{\delta u} = D_1 \frac{\delta H_n}{\delta u}$$

$n=0$ $\quad \dfrac{du}{dt_0} = 0$

$n=1$ $\quad \dfrac{du}{dt_1} = u_x$

$n=2$ $\quad \dfrac{du}{dt_2} = 6uu_x - u_{xxx}$

$n=3$ $\quad \dfrac{du}{dt_3} = \cancel{u_5 + 20u^2u_x + 10u_xu_{xx} - 10uu_{xx} - 10}$

$$= \partial_x \left(u_{xxxx} + 10u^3 - 5u_x^2 - 10uu_{xx} \right)_x$$

$$= u_5 - 10uu_{xxx} - 20u_xu_{xx} + 30u^2u_x$$

. . .

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Integrability of the KdV equation (KdV hierarchy)

$$D_2 \frac{\delta H_{n-1}}{\delta u} = D_1 \frac{\delta H_n}{\delta u} \qquad n = 0, 1, 2, \ldots$$

$$D_2 = -(D_x^3 - 4u D_x - 2u_x), \qquad D_1 = D_x$$

$n = 0$ $\quad H_0 = +\frac{1}{2}\int_{-\infty}^{\infty} u \, dx \,, \quad \frac{\delta H_0}{\delta u} = E(\frac{1}{2}u) = +1/2$

$n = 1$ $\quad H_1 = \frac{1}{2}\int_{-\infty}^{\infty} u^2 \, dx \,, \quad \frac{\delta H_1}{\delta u} = E(\frac{1}{2}u^2) = u$

$n = 2$ $\quad H_2 = \int_{-\infty}^{\infty} (u^3 + \frac{1}{2}u_x^2) \, dx \,, \quad \frac{\delta H_2}{\delta u} = E(u^3 + \frac{1}{2}u_x^2) = 3u^2 - u_{xx}$

$n = 3$ $\quad H_3 = +\int_{-\infty}^{\infty} (\frac{5}{2}u^4 + \frac{1}{2}u_{xx}^2 + 5u u_x^2) \, dx$

$$\frac{\delta H_3}{\delta u} = E(\;) = (10u^3 + 5u_x^2 - 10(uu_x)_x + u_{xxxx})$$

$n = 0$

$$D_2 \frac{\delta H_{-1}}{\delta u} = D_1 (+1/2) = 0 \quad ✓$$

$n = 1$ $\quad D_2 \frac{\delta H_0}{\delta u} = D_1(u) = u_x \quad ✓$

$n = 2$ $\quad D_2 \frac{\delta H_1}{\delta u} = D_1(3u^2 - u_{xx}) = 6uu_x - u_{xxx}$

$$= D_2(u) = -u_{xxx} + 6uu_x \quad ✓$$

Integrability of the KdV system

$$H_0 = -\frac{1}{2}\int_{-\infty}^{\infty} u(x,t)\, dx$$

$$H_1 = \frac{1}{2}\int_{-\infty}^{\infty} u^2(x,t)\, dx$$

$$H_2 = \int_{-\infty}^{\infty} dx\, \left(u^3 + \frac{1}{2} u_x^2\right)$$

$$H_{-1} = 0$$

satisfy the functional relation

$$\mathcal{D}_2 \frac{\delta H_{n-1}}{\delta u} = \mathcal{D}_1 \frac{\delta H_n}{\delta u} \qquad n = 0,1,2$$

$$\mathcal{D}_2 = D_x^3 - 4u D_x - 2u_x \quad , \quad \mathcal{D}_1 = D_x$$

$n = 0 \qquad 0 = \partial_x \left(\frac{H}{2}\right) = 0$

$n = 1 \qquad \mathcal{D}_2(-1/2) = \mathcal{D}_1 \frac{\delta H_1}{\delta u} = \mathcal{D}_1 E(\mathcal{H}_1) = \mathcal{D}_1 u$

$$= u_x \quad \checkmark$$

$n = 2 \qquad \mathcal{D}_2 \frac{\delta H_1}{\delta u} = \mathcal{D}_1 \frac{\delta H_2}{\delta u}$

$$= \mathcal{D}_2 u = \partial_x (3u^2 - u_{xx}) = u_t$$

$$\mathcal{D}_2 \frac{\delta H_{n-1}}{\delta u} = \mathcal{D}_1 \frac{\delta H_n}{\delta u}$$

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$$u_{t_n} = \mathcal{D}_2 \frac{\delta H_{n-1}}{\delta u} = \mathcal{D}_1 \frac{\delta H_n}{\delta u}$$

$n = 2 \Rightarrow$ KdV

$n = 1 \quad u_{t_1} = u_x$

$n = 0 \quad u_{t_1} = 0$

$n = 3$

$$\mathcal{D}_2 \frac{\delta H_2}{\delta u} = \mathcal{D}_1 \frac{\delta H_3}{\delta u}$$

$$\mathcal{D}_2 \, (3u^2 - u_{xx}) = -u_{xxx} + 10 u u_{xxx} + 20 u_x u_{xx} - 30 u^2 u_x$$

$$(D^3 - 4uD - 2u_x)(-u_{xx} + 3u^2)$$

$$= -u_5 + 6 (u u_x)_{xx} + 4u u_{xxx} - \cancel{2}4 u^2 u_x + 2u_x u_{xx}$$
$$- 6u^2 u_x$$

$$= -u_5 + 6(u_x^2 + u u_{xx})_x + 4u u_{xxx} - 30 u^2 u_x + 2u_x u_{xx}$$

$$= -u_5 + 20 u_x u_{xx} + 10 u u_{xxx} - 30 u^2 u_x$$

$$= \partial_x \left[-u_4 + 10 u_x^2 + 10 u u_{xx} - \cancel{5} u_x^2 - 10 u^3 \right)$$

$$= D_x \left(-u_4 + 10 u u_{xx} + 5u_x^2 - 10u^3 \right)$$

$u u_{xxx} = (u u_{xx})_x - \frac{1}{2} (u_x^2)_x \qquad E(\mathcal{H}_2) = -10u^3 + 5u_x^2 + 10 u u_{xx}$
$- u_4$

$$\mathcal{H}_3 = -\frac{5}{2}u^4 - \frac{1}{2}u_{xx}^2 + \alpha u u_x^2$$

$$\alpha u_x^2 - \alpha \partial_x 2 u u_x$$

$$\alpha u_x^2 - 2\alpha u_x^2 - 2\alpha u u_{xx}$$

$$-\alpha u_x^2 - 2\alpha u u_{xx}$$

$$\alpha = -5$$

$$5u_x^2 + 10 u u_{xx}$$

$$\mathcal{H}_3 = -\frac{5}{2}u^4 - \frac{1}{2}u_{xx}^2 - 5 u u_x^2$$

$$E(\mathcal{H}_3) = -10u^3 + u_{xxxx} - 5u_x^2 + 10(uu_x)_x$$

$$\Rightarrow H_3 = -\int_{-\infty}^{\infty}\left[\frac{5}{2}u^4 + \frac{1}{2}u_{xx}^2 + 5uu_x^2\right]dx$$

$$-10u^3 - u_4 + 5u_x^2 + 10uu_{xx}$$

$$u_{t_3} = -u_5 + 10uu_{xxx} + 2u_xu_{xx} - 30u^2u_x$$

$$\frac{dP}{dt} = \{P, H_1\}_2 = \{P, H_2\}_1$$

$$\frac{dP}{dt} = P_u u_t + P_{u_x} u_{xt} + P_{u_{xx}} u_{xxt}$$

$$= P_u u_t + P_{u_x} u_{xt} + \cdots$$

$$= P_u \{u, H_2\}_1 + P_{u_x} \partial_x \{u, H_2\}_1 + \cdots$$

$$= D \frac{\delta H_2}{\delta u}$$

$$\{P, H_2\} = \int_{-\infty}^{\infty} dx \frac{\delta P}{\delta u} D_x \frac{\delta H_2}{\delta u}$$

$$= \int_{-\infty}^{\infty} dx \frac{\delta P}{\delta u} u_t$$

$$= \int_{-\infty}^{\infty} dx\, E(\mathcal{P})\, u_t$$

$$= \int_{-\infty}^{\infty} dx\, u_t \left(\frac{\partial \mathcal{P}}{\partial u} - D \frac{\partial \mathcal{P}}{\partial u_x} + \cdots \right)$$

$$= \int_{-\infty}^{\infty} dx \left(\frac{\partial \mathcal{P}}{\partial u} u_t + \frac{\partial \mathcal{P}}{\partial u_x} u_{tx} + \cdots \right)$$

$$= \int_{-\infty}^{\infty} dx \frac{d}{dt} \mathcal{P} = \frac{d}{dt} P.$$

Hence we have

$$\frac{dP}{dt} = \{P, \cancel{H}_2\}_1 = \{\}$$

if P is a conserved quantity
then $\{P, H_2\}_1 = 0$.

all H_n are conserved quantit.

$$\frac{dH_n}{dt} = \{H_n, H_2\}_1$$

$$= \int_{-\infty}^{\infty} \frac{\delta H_n}{\delta u} D_x \frac{\delta H_2}{\delta u}$$

$$= \int_{-\infty}^{\infty} \frac{\delta H_n}{\delta u} (3u^2 - u_{xx})_x \, dx$$

$$= \int_{-\infty}^{\infty} \frac{\delta H_n}{\delta u} u_t \, dx = \int_{-\infty}^{\infty} \frac{\delta H_n}{\delta u} D_2 \frac{\delta H_1}{\delta u}$$

$$\{H_n, H_2\}_1 = \{H_n, H_1\}_2$$

$$= \int_{-\infty}^{\infty} \frac{\delta H_n}{\delta u} (D^3 - 4uD - 2u_x) u$$

$$= \int_{-\infty}^{\infty} \frac{\delta H_n}{\delta u} D_2 \frac{\delta H_1}{\delta u} dx$$

$$= - \int_{-\infty}^{\infty} D_2 \frac{\delta H_n}{\delta u} \frac{\delta H_1}{\delta u}$$

$$= -\int_{-\infty}^{\infty} D_1 \frac{\delta H_{n+1}}{\delta u} \quad \frac{\delta H_1}{\delta u}$$

$$= \int_{-\infty}^{\infty} \frac{\delta H_{n+1}}{\delta u} \; D_1 \; \frac{\delta H_1}{\delta u} \quad = \{ H_{n+1}, H_1 \}_1$$

$$\frac{d H_2}{dt} = 0 \quad , \quad \frac{d H_1}{dt} = 0$$

$$\{ H_m, H_n \}_1 = \int \frac{\delta H_m}{\delta u} \, D_x \, \frac{\delta H_n}{\delta u} \, dx$$

$$= + \int \frac{\delta H_m}{\delta u} \, D_2 \, \frac{\delta H_{n-1}}{\delta u} \, dx$$

$$= - \int \quad D_2 \frac{\delta H_m}{\delta u} \quad \frac{\delta H_{n-1}}{\delta u}$$

$$= - \int \quad D_x \; \frac{\delta H_{m+1}}{\delta u} \quad \frac{\delta H_{n-1}}{\delta u}$$

$$= \quad \{ H_{m+1}, H_{n-1} \}$$

$$\{ H_m, H_n \}_1 = \{ H_{m+1}, H_{n-1} \}_1$$

$m = 1, \quad n = 2$

$$\{ H_1, H_2 \}_1 = \{ H_2, H_1 \} = 0$$

$m = 2, \quad n = 3$

$$\{ H_1, H_3 \}_1 = \{ H_3, H_2 \}_1$$

$m = 1, \quad n = 2$

$$\{ H_2, H_2 \} = \{ H_3, H_1 \} = 0$$

$\frac{dH_3}{dt} = 0$ ✓ iteratively one can prove that

$$\{ H_n, H_1 \} = 0 \Rightarrow \frac{dH_n}{dt} = 0$$

Lemma 1.

$$\{P, H_2\}_1 = \int_{-\infty}^{\infty} dx \frac{\delta P}{\delta u(x)} \mathcal{D}_1 \frac{\delta H_2}{\delta u}$$

$$= \int_{-\infty}^{\infty} dx \frac{\delta P}{\delta u(x)} u_t$$

$$= \int_{-\infty}^{\infty} dx\, E(\mathcal{P})\, u_t$$

$$= \int_{-\infty}^{\infty} dx \left[\frac{\partial \mathcal{P}}{\partial u} - \partial_x \frac{\partial \mathcal{P}}{\partial u_x} + \cdots \right] u_t$$

$$= \int_{-\infty}^{\infty} dx \left[\frac{\partial \mathcal{P}}{\partial u} + \frac{\partial \mathcal{P}}{\partial u_x} u_{xt} + \cdots \right]$$

$$= \int_{-\infty}^{\infty} dx \frac{d}{dt} \mathcal{P} = \frac{d}{dt} P$$

$$\frac{dP}{dt} = \{P, H_2\}_1 = \{P, H_1\}_2$$

<u>corollary</u>

if $P = H_1 \Rightarrow \frac{dH_1}{dt} = 0$

if $P = H_2 \Rightarrow \frac{dH_2}{dt} = 0$

Corollary:

a) $P = H_1 \Rightarrow \frac{dH_1}{dt} = 0 \qquad \frac{dH_1}{dt} = \{H_1, H_1\}_2 = 0$

b) $P = H_2 \Rightarrow \frac{dH_2}{dt} = 0 \qquad \frac{dH_2}{dt} = \{H_2\, H_2\}_1 = 0$

$$\{H_m, H_n\}_1 = \int_{-\infty}^{\infty} \frac{\delta H_m}{\delta u} D_* \frac{\delta H_n}{\delta u}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{\delta H_m}{\delta u} D_2 \frac{\delta H_{n-1}}{\delta u}\, dx$$

$$= -\int_{-\infty}^{\infty} D_2 \frac{\delta H_m}{\delta u} \frac{\delta H_{n-1}}{\delta u}$$

$$= -\int_{-\infty}^{\infty} D_1 \frac{\delta H_{m+1}}{\delta u} \frac{\delta H_{n-1}}{\delta u}$$

$$= \int \frac{\delta H_{m+1}}{\delta u} D_1 \frac{\delta H_{n-1}}{\delta u}$$

$$= \{H_{m+1}, H_{n-1}\}_1$$

Lemma 2.

$$\mathcal{D}_2 \frac{\delta H_n}{\delta u} = \mathcal{D}_1 \frac{\delta H_{n+1}}{\delta u}$$

$$\{H_n, H_m\}_1 = \int \frac{\delta H_n}{\delta u} \underbrace{D_x \frac{\delta H_m}{\delta u}} dx$$

$$= \int \frac{\delta H_n}{\delta u} D_2 \frac{\delta H_{m-1}}{\delta u}$$

$$= - \int \frac{\delta H_{m-1}}{\delta u} D_2 \frac{\delta H_n}{\delta u}$$

$$= - \int \frac{\delta H_{m-1}}{\delta u} \mathcal{D}_1 \frac{\delta H_{n+1}}{\delta u}$$

$$= - \{H_{m-1}, H_{n+1}\}_1$$

$$\{H_n, H_m\}_1 = \{H_{n+1}, H_{m-1}\},$$

$$\{H_{\cancel{2}}, H_2\} = \{H_3, H_1\}_1$$

$$\{H_n, H_2\}_1 = \{H_{n+1}, H_1\}_1$$

i) $n=1$ $\quad \{H_1, H_2\}_1 = \{H_2, H_1\}_1 \Rightarrow$

i) $n=2$ $\quad \{H_2, H_2\}_1 = \{H_3, H_1\}_1 \Rightarrow$

i) $n=3$ $\quad \{H_3, H_2\}_1 = \{H_4, H_1\}_1 = \{H_5, H_0\}$

$$\{H_n, H_1\}_1 = \{H_{n+1}, H_0\}$$

$$\{H_{n+1}, H_{n-1}\}_1 = 0$$

$$\{H_n, H_m\}_1 = \{H_{n+2}, H_{m-2}\}$$

$$\{H_2, H_3\}_1 = \{H_3, H_2\}_1$$

$$\Rightarrow \{H_3, H_2\}_1 = 0$$

$$\{H_2, H_3\}_1 = \{H_3, H_2\}_1 = 0$$

$$\frac{dH_3}{dt} = 0$$

$$\{H_n, H_4\}_1 = \{H_{n+1}, H_3\}_1$$

$$\{H_2, H_4\} = 0$$

$$\frac{dH_4}{dt} = 0 \quad \checkmark \qquad \{H_n, H_m\} = \{H_{n+k}, H_{m-k}\}$$

$$m - k = 2$$

$$\{H_2, H_n\}_1 = 0 \qquad \{H_n, H_m\}_1 = \{H_{n+m-2}, H_2\}$$

$$\{H_n, H_2\}_1 = -\{H_1, H_{n+1}\} \quad = \frac{dH_{n+m-2}}{dt}$$

$$= -\{H_2, H_n\}_1$$

Lemma: $\{H_n, H_m\}_1 = 0 \qquad m = 1, 2, \ldots, n-1, n$

proof: We have, from page (20)

$$\{H_m, H_n\}_1 = \{H_{m+1}, H_{n-1}\}_1$$

i) let $m = n-1 \Rightarrow \{H_{n-1}, H_n\}_1 = \{H_n, H_{n-1}\}$

$$\Rightarrow \{H_n, H_{n-1}\}_1 = 0$$

ii) $m = n-2 \Rightarrow \{H_{n-2}, H_n\} = \{H_{n-1}, H_{n-1}\} = 0$

$$\{H_n, H_{n-2}\}_1 = 0$$

iii) $m = n-3 \Rightarrow \{H_{n-3}, H_n\}_1 = \{H_{n-2}, H_{n-1}\}_1 = 0$

since $\{H_n, H_{n-1}\}_1 = 0 \Rightarrow \{H_{n-1}, H_{n-2}\}_1 = 0$

$$\Rightarrow \{H_n, H_{n-3}\}_1 = 0$$

iv) $m = n-k \quad , \quad k \leq n$

$$\{H_{n-k}, H_n\}_1 = \{H_{n+1-k}, H_{n-1}\}_1$$

$\{H_n, H_{n-(k-1)}\}_1 = 0$ by the previous calculations

$$\Rightarrow \{H_{n-k}, H_n\} = 0$$

$$\Rightarrow \{H_n, H_m\}_1 = 0 \qquad m = 1, 2, \ldots, n$$

$$H_0 = -\frac{1}{2}\int_{-\infty}^{\infty} u\,dx \qquad \frac{\delta H_0}{\delta u} = 1$$

$$H_1 = \frac{1}{2}\int_{-\infty}^{\infty} u^2\,dx \qquad \frac{\delta H_1}{\delta u} = u$$

$$H_2 = \int_{-\infty}^{\infty} [u^3 + \tfrac{1}{2}u_x^2] \qquad \frac{\delta H_2}{\delta u} = 3u^2 - u_{xx}$$

$$H_3 = 5u^4 + 10uu_x^2 + u_{xx}^2$$

$$H_4 = 21u^5 + 105u^2u_x^2 + 21uu_{xx}^2 + u_3^2$$

$$\mathcal{D}_1 \frac{\delta H_{n+1}}{\delta u} = \mathcal{D}_2 \frac{\delta H_n}{\delta u}$$

$$\mathcal{D}_1 = \partial_x \,, \quad \mathcal{D}_2 = -\partial^3 + 4u\partial + 2u_x$$

$$\boxed{(-1)^{n+1}\frac{\partial u}{\partial t_n} = \mathcal{D}_1 \frac{\delta H_{n+1}}{\delta u} = \mathcal{D}_2 \frac{\delta H_n}{\delta u}} \qquad n = 1, 2, \ldots$$

Recursion operator $\quad R = \mathcal{D}_2 \cdot \mathcal{D}_1^{-1}$

we had the hierarchy

$$u_{t_N} \mp \mathcal{R}^N u_x = 0$$

$$u_{t_2} + 30u^2u_x - 20u_xu_{xx} - 10uu_{xxx} + u_{xxxxx} = 0$$

$$\frac{\partial u}{\partial t_2} = \mathcal{D}_2 \frac{\delta H_2}{\delta u} = \mathcal{D}_2 (3u^2 - u_{xx})$$

Sign difference

1) conserved quantities

proof of $\{H_n, H_m\}_1 = 0$ for all m, n

$\{H_n, H_m\}_2 = 0$

we showed that

$$\{H_n, H_2\}_1 = 0 \Rightarrow \frac{dH_n}{dt} = 0$$

prove first $\{H_n, H_1\}_2 = 0 \Rightarrow \frac{dH_n}{dt} = 0$

$$\partial_x (u_4 + \dots) = (-\partial_x^3 + \quad)(-\partial_{xx} \dots)$$

$$= u_5 + \dots$$

$$\partial_x (-u_6 \quad) = -\partial_x^3 (\quad$$

2. AKNS FORMULATION

$$\begin{pmatrix} p \\ q \end{pmatrix}_t = 2\sigma_3 J^{N-1} \begin{pmatrix} p_x \\ -q_x \end{pmatrix} = 2\sigma_3 J^{N-1} \sigma_3 \begin{pmatrix} p_x \\ q_x \end{pmatrix}$$

$$= 2\sigma_3 J \sigma_3 \sigma_3 J \dots J \sigma_3$$

$$\sigma_3 J \sigma_3 = \begin{pmatrix} -pD^{-1}q + \frac{1}{2}D & pD^{-1}p \\ qD^{-1}q & -qD^{-1}q + \frac{1}{2}D \end{pmatrix} \sigma_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -pD^{-1}q + \frac{1}{2}D & -pD^{-1}p \\ qD^{-1}q & qD^{-1}p - \frac{1}{2}D \end{pmatrix}$$

This is called the recursion operator

$$R = \begin{pmatrix} -pD^{-1}q + \frac{1}{2}D & -pD^{-1}p \\ qD^{-1}q & +qD^{-1}p - \frac{1}{2}D \end{pmatrix}$$

$$\begin{pmatrix} p \\ q \end{pmatrix}_{t_{N-1}} = R^{N-1} \begin{pmatrix} p_x \\ q_x \end{pmatrix}$$

AKNS Formulation

To obtain more integrable system, from the Lax equations AKNS introduced the following formulation. Let ψ be a column matrix then the Lax equations are defined as

$$\frac{\partial \psi}{\partial t} = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}\psi \,, \quad \frac{\partial \psi}{\partial x} = \begin{pmatrix} \lambda & p \\ q & -\lambda \end{pmatrix}\psi \qquad (1)$$

Where $A = A(x,t;\lambda)$, $B = B(x,t;\lambda)$, $C = C(x,t;\lambda)$ and $p = p(x,t)$, $q = q(x,t)$ and λ is the spectral parameter (constant). The integrability condition $\psi_{xt} = \psi_{tx}$ gives us.

$$\begin{pmatrix} A & B \\ C & -A \end{pmatrix}_x + \begin{pmatrix} A & B \\ C & -A \end{pmatrix}\begin{pmatrix} \lambda & p \\ q & -\lambda \end{pmatrix} = \begin{pmatrix} \lambda & p \\ q & -\lambda \end{pmatrix}_t + \begin{pmatrix} \lambda & p \\ q & -\lambda \end{pmatrix}\begin{pmatrix} A & B \\ C & -A \end{pmatrix} \qquad (2)$$

or

$$A_x = pC - qB \quad — \quad (3)$$

$$q_t = 2\lambda C - 2Aq + C_x \quad — \quad (4)$$

$$p_t = -2\lambda B + 2pA + B_x \quad — \quad (5)$$

Let us assume that A, B, C are analytic in λ, i.e

$$A = \sum_{n=0}^{N}{}' a_{N-n}\lambda^n , \quad B = \sum_{n=0}^{\infty}{}' b_{N-n}\lambda^n , \quad C = \sum_{n=0}^{\infty}{}' c_{N-n}\lambda^n \tag{6}$$

using (6) in (3-5) we get

λ^0-terms in (4-5)

$$q_t = -2q A_N + C_{N,x} , \tag{7}$$

$$p_t = 2p A_N + B_{N,x} , \tag{8}$$

and letting $m = N-n$ in (3,4,5)

$$2C_{m+1} = 2q A_m - C_{m,x} \tag{9}$$

$$2 b_{m+1} = 2p A_m + b_{m,x} \tag{10}$$

$$a_m = D^{-1}(p c_m) - D^{-1}(q b_m) , \tag{11}$$

$$D^{-1} = \int^x$$

$m = 0, 1, \dots, N-1$.

or

$$b_{m+1} = q D^{-1}(p c_m) - p D^{-1}(q b_m) + \tfrac{1}{2} b_{m,x} , \tag{12}$$

$$c_{m+1} = q D^{-1}(p c_m) - q D^{-1}(q b_m) - \tfrac{1}{2} c_{m,x} , \tag{13}$$

$$\begin{pmatrix} b_{m+1} \\ c_{m+1} \end{pmatrix} = \begin{pmatrix} \tfrac{1}{2}D - pD^{-1}q & pD^{-1}p \\ -qD^{-1}q & qD^{-1}p - \tfrac{1}{2}D \end{pmatrix} \begin{pmatrix} b_m \\ c_m \end{pmatrix} \tag{14}$$

Let

$$J=\begin{pmatrix}-pD^{-1}q+\frac{1}{2}D & pD^{-1}p\\ -qD^{-1}q & qD^{-1}p-\frac{1}{2}D\end{pmatrix} \qquad (15).$$

$$\Rightarrow \begin{pmatrix}b_{m+1}\\ c_{m+1}\end{pmatrix}=J\begin{pmatrix}b_m\\ c_m\end{pmatrix}=J^2\begin{pmatrix}b_{m-1}\\ c_{m-1}\end{pmatrix}=J^{m+1}\begin{pmatrix}b_0\\ c_0\end{pmatrix}$$

$$m=0,1,2,\cdots,N-1 \qquad (16)$$

on the other hand

$$q_t=-2qA_N+C_{N,x}=-2q(D^{-1}pC_N-D^{-1}qB_N)+C_{N,x}$$

$$p_t=2pA_N+B_{N,x}=2p(D^{-1}pC_N-D^{-1}qB_N)+B_{N,x}$$

$$\begin{pmatrix}p\\ q\end{pmatrix}_t=2\begin{pmatrix}-pD^{-1}q+\frac{1}{2}D & pD^{-1}p\\ qD^{-1}p & -qD^{-1}p+\frac{1}{2}D\end{pmatrix}\begin{pmatrix}b_N\\ c_N\end{pmatrix} \qquad (17)$$

$$\sigma_3 J=\begin{pmatrix}1&0\\0&-1\end{pmatrix}J=\begin{pmatrix}-pD^{-1}q+\frac{1}{2}D & pD^{-1}p\\ qD^{-1}q & -qD^{-1}p+\frac{1}{2}D\end{pmatrix} \qquad (18)$$

Hence

$$\begin{pmatrix}p\\ q\end{pmatrix}_t=2\sigma_3 J\begin{pmatrix}b_N\\ c_N\end{pmatrix} \qquad (19)$$

We know that, from (16), let $m = N-1$

$$\begin{pmatrix} b_N \\ c_N \end{pmatrix} = J^N \begin{pmatrix} b_0 \\ c_0 \end{pmatrix}$$

Hence

$$\begin{pmatrix} p \\ t \end{pmatrix}_{t_N} = 2\sigma_3 J^{N+1} \begin{pmatrix} b_0 \\ c_0 \end{pmatrix}, \quad J = \begin{pmatrix} -pD^{-1}q + \frac{1}{2}D & pD^{-1}p \\ -qD^{-1}q & qD^{-1}p - \frac{1}{2}D \end{pmatrix}$$

i) $b_0 = c_0 = 0 \qquad (N=0)$

$$\begin{pmatrix} p \\ q \end{pmatrix}_{t_0} = 2\sigma_3 J \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 2\sigma_3 \begin{pmatrix} p\, D^{-1}(c_0 p - b_0 q) \\ q\, D^{-1}(c_0 p - b_0 q) \end{pmatrix} = 2\alpha\sigma_3 \begin{pmatrix} p \\ q \end{pmatrix}$$

$$= 2\alpha \begin{pmatrix} p \\ -q \end{pmatrix}$$

$$p_{t_0} = 2\alpha p, \quad q_{t_0} = -2\alpha q$$

ii) $N = 1$

$$\begin{pmatrix} p \\ q \end{pmatrix}_{t_1} = 2\sigma_3 J^2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 2\sigma_3 J \; J\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 2\alpha\sigma_3 J \begin{pmatrix} p \\ q \end{pmatrix}$$

$$= 2\alpha\sigma_3 \begin{pmatrix} \frac{1}{2} p_x \\ -\frac{1}{2} q_x \end{pmatrix} = \alpha \begin{pmatrix} p_x \\ q_x \end{pmatrix}$$

$$p_{t_1} = \alpha p_x, \quad q_{t_2} = \alpha q_x$$

$N=2$

$$\begin{pmatrix} p \\ q \end{pmatrix}_{t_2} = 2\sigma_3 J^2 J \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 2\sigma_3 J J^2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 2\sigma_3 J \alpha \begin{pmatrix} \frac{1}{2} p_x \\ -\frac{1}{2} q_x \end{pmatrix}$$

$$= \alpha \sigma_3 J \begin{pmatrix} p_x \\ -q_x \end{pmatrix}$$

$$= \alpha \sigma_3 \begin{pmatrix} -pD^{-1} q p_x + \frac{1}{2} p_{xx} - pD^{-1} p q_x \\ -qD^{-1} q p_x - qD^{-1} p q_x + \frac{1}{2} q_{xx} \end{pmatrix}$$

$$= \alpha \sigma_3 \begin{pmatrix} -p^2 q + \frac{1}{2} p_{xx} \\ -q^2 p + \frac{1}{2} q_{xx} \end{pmatrix}$$

$$p_{t_2} = \alpha \left(\frac{1}{2} p_{xx} - p^2 q \right), \quad q_{t_2} = \alpha \left(-\frac{1}{2} q_{xx} + q^2 p \right)$$

$N=3$.

$$\begin{pmatrix} p \\ q \end{pmatrix}_{t_3} = 2\sigma_3 J J^3 \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 2\sigma_3 J \begin{pmatrix} -p^2 q + \frac{1}{2} p_{xx} \\ -q^2 p + \frac{1}{2} q_{xx} \end{pmatrix}$$

$$= \frac{\alpha}{2} \begin{pmatrix} \frac{1}{2} p_{xxx} - 3pq p_x \\ \frac{1}{2} q_{xxx} + 3qp q_x \end{pmatrix}$$

$$p_{t_3} = \frac{\alpha}{4} \left(p_{xxx} - 6pq p_x \right)$$

$$q_{t_3} = \frac{\alpha}{4} \left(q_{xxx} - 6pq q_x \right)$$

$p=q$ corresponds to the KdV equation

$$b_0 = c_0 = 0$$

$$\begin{pmatrix} b_1 \\ c_1 \end{pmatrix} = J \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} p \\ q \end{pmatrix}$$

$$\begin{pmatrix} b_2 \\ c_2 \end{pmatrix} = J^2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \alpha J \begin{pmatrix} p \\ q \end{pmatrix} = \frac{\alpha}{2} \begin{pmatrix} p_x \\ q_x \end{pmatrix}$$

$$\begin{pmatrix} b_3 \\ c_3 \end{pmatrix} = \frac{\alpha}{2} \begin{pmatrix} -p^2 q + \frac{1}{2} p_{xx} \\ -q^2 p + \frac{1}{2} q_{xx} \end{pmatrix}$$

$$\begin{pmatrix} b_4 \\ c_4 \end{pmatrix} = \frac{\alpha}{4} \begin{pmatrix} \frac{1}{2} p_{xxx} - 3pq\, p_x \\ -\frac{1}{2} q_{xxx} + 3qp\, q_x \end{pmatrix}$$

– – – –

$$\begin{pmatrix} b_{m+1} \\ c_{m+1} \end{pmatrix} = J^{m+1} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad m = 0, 1, 2, \dots, N-1$$

$$A_0 = \text{constn} = \alpha$$

$$A_1 = D^{-1} p \alpha q - D^{-1} q \alpha p = 0$$

$$A_2 = D^{-1}\left(-\frac{\alpha}{2} q_x p\right) - D^{-1} \frac{\alpha}{2} q p_x = -\frac{\alpha}{2} pq$$

$$A_3 = D^{-1} p c_3 - D^{-1} q b_3$$

$$= \frac{\alpha}{2} D^{-1} p \left(-q^2 p + \frac{1}{2} q_{xx}\right) - \frac{\alpha}{2} D^{-1} q \left(-p^2 q + \frac{1}{2} p_{xx}\right)$$

$$= \frac{\alpha}{4} (p q_x - q p_x)$$

$$A_y = D^{-1} p c_y - D^{-1} q b_y$$

$$= \frac{\alpha}{4} D^{-1} p\left(-\frac{1}{2} q_{xxx} + 3pqq_x\right) - \frac{\alpha}{4} D^{-1} q \left(\frac{1}{2} p_{xxx} - 3pqp_x\right)$$

$$= \frac{\alpha}{8} D^{-1}\left(-p q_{xxx} - q p_{xxx}\right) + \frac{3\alpha}{4} D^{-1} p^2 q q_x + \frac{3\alpha}{4} D^{-1} q^2 p p_x$$

$$= -\frac{\alpha}{8}\left(D^{-1}\left[(p q_{xx} + q p_{xx})_x\right] - D^{-1}(p_x q_{xx} + q_x p_{xx})\right)$$

$$+ \frac{3\alpha}{4} p^2 q^2$$

$$= -\frac{\alpha}{3}\left(p q_{xx} + q p_{xx}\right) + \frac{\alpha}{8}\left(p_x q_x\right) + \frac{3\alpha}{4} p^2 q^2$$

$$p_{t_1} = p_x$$
$$q_{t_1} = q_x$$

$$p_{t_2} = p_{xx} - 2p^2 q$$
$$q_{t_2} = -q_{xx} + 2q^2 p$$

$$p_{t_3} = p_{xxx} - 6pqp_x$$
$$q_{t_3} = q_{xxx} - 6pqq_x$$

$$\begin{pmatrix} p \\ q \end{pmatrix}\Bigg|_{t_{N-1}} = \mathbb{R}^{N-1}\begin{pmatrix} p_x \\ q_x \end{pmatrix}$$

Let

$$D_2 = \begin{pmatrix} pD^{-1}p & \frac{1}{2}D - pD^{-1}q \\ \frac{1}{2}D - qD^{-1}p & qD^{-1}q \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\checkmark \quad D_2 D_1^{-1} = -D_2 D_1 = \begin{pmatrix} \frac{1}{2}D - pD^{-1}q & -pD^{-1}p \\ +qD^{-1}q & -\frac{1}{2}D + qD^{-1}p \end{pmatrix}$$

$$D_1^{-1} D_2 = -D_1 D_2 = -\begin{pmatrix} \frac{1}{2}D - pD^{-1}q & qD^{-1}q \\ -pD^{-1}p & -\frac{1}{2}D + pD^{-1}q \end{pmatrix}$$

$$D_2 D_1^{-1} = \mathbb{R} \quad \checkmark$$

D_2 and D_1 are the Hamilton operators of the AKNS system

$$H_0 = \alpha \int p q \, dx \quad , \quad \mathcal{H}_0 = \alpha p q$$

$$H_1 = -\frac{\alpha}{4} \int (p q_x - q p_x) \, dx \quad , \quad \mathcal{H}_1 = -\frac{\alpha}{4} (p q_x - q p_x)$$

$$H_2 = \frac{\alpha}{8} \int \left(p q_{xx} + q p_{xx} - 2 p^2 q^2 \right) dx \quad , \quad \mathcal{H}_2 = \frac{\alpha}{8} \left(p q_{xx} + q p_{xx} - 2 p^2 q^2 \right)$$

$$u_{t_n} = D_1 \frac{\delta H_n}{\delta u} = D_2 \frac{\delta H_{n-1}}{\delta u} \quad , \quad n = 0, 1, 2$$

$$\frac{\delta H}{\delta u} = \begin{pmatrix} \frac{\delta H}{\delta p} \\ \frac{\delta H}{\delta q} \end{pmatrix} = \begin{pmatrix} E_p(\mathcal{H}) \\ E_q(\mathcal{H}) \end{pmatrix}$$

$$\frac{\delta H_0}{\delta u} = \begin{pmatrix} \alpha q \\ \alpha p \end{pmatrix} \qquad u_{t_0} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha q \\ \alpha p \end{pmatrix} = \begin{pmatrix} \alpha p \\ -\alpha q \end{pmatrix}$$

$$p_{t_0} = \alpha p \quad , \quad q_{t_0} = -\alpha q$$

$$H_{-1} = 0 \qquad D_2 (0) = D_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha p \\ -\alpha q \end{pmatrix} \checkmark$$

$n=1$

$$u_{t_1} = D_1 \frac{\delta H_1}{\delta u} = D_2 \frac{\delta H_0}{\delta u}$$

$$D_2 \frac{\delta H_0}{\delta u} = \alpha\, D_2 \begin{pmatrix} q \\ p \end{pmatrix} = \alpha \begin{pmatrix} \cancel{pD^{-1}pq} + \frac{1}{2}p_x - \cancel{pD^{-1}qp} \\ \frac{1}{2}q_x - \cancel{pD^{-1}pq} + \cancel{qD^{-1}qp} \end{pmatrix} = \frac{\alpha}{2} \begin{pmatrix} p_x \\ q_x \end{pmatrix}$$

$$p_{t_1} = \frac{\alpha}{2} p_x, \quad q_{t_1} = \frac{\alpha}{2} q_x$$

$$D_1 \frac{\delta H_1}{\delta u} = D_1 E(H_1) = D_1 \begin{pmatrix} E_p(H_1) \\ E_q(H_1) \end{pmatrix} \cancel{= D_1}$$

$$= -\frac{\alpha}{4} D_1 \begin{pmatrix} 2q_x \\ -2p_x \end{pmatrix} = -\frac{\alpha}{2} D_1 \begin{pmatrix} q_x \\ -p_x \end{pmatrix} = -\frac{\alpha}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} q_x \\ -p_x \end{pmatrix}$$

$$= -\frac{\alpha}{2} \begin{pmatrix} -p_x \\ -q_x \end{pmatrix} = \frac{\alpha}{2} \begin{pmatrix} p_x \\ q_x \end{pmatrix} \checkmark$$

$n=2$

$$u_{t_2} = D_1 \frac{\delta H_2}{\delta u} = D_2 \frac{\delta H_1}{\delta u}$$

$$D_2 \frac{\delta H_1}{\delta u} = D_2 \begin{pmatrix} E_p(H_1) \\ E_q(H_1) \end{pmatrix} = \cancel{D_2} - \frac{\alpha}{4} D_2 \begin{pmatrix} 2q_x \\ -2p_x \end{pmatrix}$$

$$= -\frac{\alpha}{2} D_2 \begin{pmatrix} q_x \\ -p_x \end{pmatrix} = -\frac{\alpha}{2} \begin{pmatrix} pD^{-1}(pq_x + qp_x) - p_{xx} \\ qD^{-1}(pq_x + qp_x) + q_{xx} \end{pmatrix}$$

$$= +\frac{\alpha}{2} \begin{pmatrix} p_{xx} - p^2 q \\ -q_{xx} + q^2 p \end{pmatrix}$$

$$R = \begin{pmatrix} -pD^{-1}q + \frac{1}{2}D & -pD^{-1}p \\ qD^{-1}q & qD^{-1}p - \frac{1}{2}D \end{pmatrix}$$

$$R\begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} -pD^{-1}qp_x + \frac{1}{2}p_{xx} - pD^{-1}pq_x \\ qD^{-1}qp_x + qD^{-1}pq_x - \frac{1}{2}q_{xx} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}p_{xx} - p^2 q \\ -\frac{1}{2}q_{xx} + q^2 p \end{pmatrix}$$

$$p_{t_1} = \frac{1}{2}p_{xx} - p^2 q$$

$$q_{t_1} = -\frac{1}{2}q_{xx} + q^2 p$$

$$R\begin{pmatrix} \frac{1}{2}p_{xx} - p^2 q \\ -\frac{1}{2}q_{xx} + q^2 p \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}pD^{-1}qp_{xx} + \cancel{pD^{-1}p^2q^2} + \frac{1}{4}p_{xxx} - \frac{1}{2}(p^2q)_x \\ +\frac{1}{2}pD^{-1}pq_{xx} - \cancel{pD^{-1}p^2q^2} \\ \frac{1}{2}qD^{-1}qp_{xx} - \cancel{\frac{1}{2}(p^2q)_x} \quad \cancel{+\frac{1}{2}pD^{-1}pq_{xx} - pD^{-1}} \\ \cancel{-\frac{1}{2}pD^{-1}p^2q^2} \\ -\frac{1}{2}qD^{-1}pq_{xx} + \cancel{qD^{-1}q^2p^2} \\ +\frac{1}{4}q_{xxx} - \frac{1}{2}(q^2p)_x \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4}p_{xxx} + \frac{1}{2}p(\cancel{p}q_x - qp_x) - \frac{1}{2}(2pqp_x + \cancel{p^2q_x}) \\ \frac{1}{4}q_{xxx} + \frac{1}{2}q(\cancel{q}p_x - pq_x) - \frac{1}{2}(2qpq_x + \cancel{q^2p_x}) \end{pmatrix}$$

$$p_{t_2} = \frac{1}{4} p_{xxx} - \frac{3}{2} p q p_x$$

$$q_{t_2} = \frac{1}{4} q_{xxx} - \frac{3}{2} p q q_x$$

i) $q = q_0$ constant

$$p_t = \frac{1}{4} p_{xxx} - \frac{3}{2} q_0 \, p p_x \qquad (KdV)$$

ii) $p = q$

$$p_t = \frac{1}{4} p - \frac{3}{2} p^2 p_x \qquad (mKdV)$$

$$u_t - R u_x = R^{-1} u_y$$

$$R(u_t - R u_x) = u_y$$

Generation of the AKNS hierarchy.

$$u_{t_N} = \alpha_N \mathcal{R}^{N-1} u_x \quad , \quad u = \begin{pmatrix} p \\ q \end{pmatrix}$$

1) $N = 1$

$$p_{t_1} = \alpha_1 p_x \quad , \quad q_{t_1} = \alpha_1 q_x$$ (simple wave eqns).

2) $N = 2$

$$p_{t_2} = \frac{\alpha_2}{2}\left(p_{xx} - 2p^2 q\right)$$
$$q_{t_2} = \frac{\alpha_2}{2}\left(q_{xx} - 2q^2 p\right)$$

3) $N = 3$

$$p_{t_3} = \frac{\alpha_3}{4}\left(p_{xxx} - 6pq\,p_x\right)$$
$$q_{t_3} = \frac{\alpha_3}{4}\left(q_{xxx} - 6pq\,q_x\right)$$

$q = q_0 \Rightarrow \quad p_{t_3} = \frac{\alpha_3}{4}\left(p_{xxx} - 6q_0\, p p_x\right)$ (KdV)

$p = q \Rightarrow \quad p_{t_3} = \frac{\alpha_3}{4}\left(p_{xxx} - 6p^2 p_x\right)$ (mKdV)

4) $N = 4$ ----

Lax Formulation with Poisson Bracket

Let $f(x,p)$ and $g(x,p)$ be differentiable functions. Then the standard Poisson bracket is defined by

$$\{f,g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p}$$

This baracket satisfies

a) skew symmetric : $\{f,g\} = -\{g,f\}$

b) Jacobi identity : $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$

proof: a) $\{f,g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} = -\left(\frac{\partial g}{\partial p}\frac{\partial f}{\partial x} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}\right)$

$= -\{g,f\}$

b) $\{f,\{g,h\}\} = \frac{\partial f}{\partial p}\frac{\partial}{\partial x}\{g,h\} - \frac{\partial f}{\partial x}\frac{\partial}{\partial p}\{g,h\}$

$$\{f,\{g,h\}\} = f_p \frac{\partial}{\partial x}(g_p h_x - g_x h_p) - f_x \frac{\partial}{\partial p}(g_p h_x - g_x h_p)$$

$$= f_p(\underbrace{g_{px} h_x}_{4} + \underbrace{g_p h_{xx}}_{7} - \underbrace{g_{xx} h_p}_{8} - \underbrace{g_x h_{px}}_{5}) - f_x(\underbrace{g_{pp} h_x}_{1} + \underbrace{g_p h_{xp}}_{5} - \underbrace{g_{xp} h_p}_{4} - \underbrace{g_x h_{pp}}_{3})$$

$f \to g,\ g \to h,\ h \to f$

$$\{g,\{h,f\}\} = g_p(\underbrace{h_{px} f_x}_{5} + \underbrace{h_p f_{xx}}_{9} - \underbrace{h_{xx} f_p}_{7} - \underbrace{h_x f_{px}}_{6}) - g_x(\underbrace{h_{pp} f_x}_{3} + \underbrace{h_p f_{xp}}_{6} - \underbrace{h_{xp} f_p}_{5} - \underbrace{h_x f_{pp}}_{2})$$

$$\{h,\{f,g\}\} = h_p(\underbrace{f_{px} g_x}_{6} + \underbrace{f_p g_{xx}}_{8} - \underbrace{f_{xx} g_p}_{9} - \underbrace{f_x g_{px}}_{4}) - h_x(\underbrace{f_{pp} g_x}_{2} + \underbrace{f_x g_{xp}}_{4} - \underbrace{f_{xp} g_p}_{6} - \underbrace{f_x g_{pp}}_{1})$$

$\Rightarrow \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$

We shall use a modification of the standard Poisson bracket

$$\{f,g\}_k = p^k \{f,g\}$$

where k is an integer. This new bracket also defines a Poisson bracket.

proof:

a) $\{f,g\}_k = p^k \{f,g\} = -p^k \{g,f\}$

$= -\{g,f\}_k.$

b) $\{f,\{g,h\}_k\}_k = p^k (f_p \, (p^k \{g,h\})_x$

$- p^k (f_x \, (p^k \{g,h\})_p)$

$= p^{2k} (f_p \{g,h\}_x - f_x \{g,h\}_p)$

$- k\, p^{2k-1} f_x \{g,h\}$

$\Rightarrow$

$\{f,\{g,h\}_k\}_k + \{g,\{h,f\}_k\}_k + \{h,\{f,g\}_k\}_k$

$= p^{2k} [\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\}]$

$- k\, p^{2k-1} [f_x \{g,h\} + g_x \{h,f\} + h_x \{f,g\}]$

$= -k\, p^{2k-1} [f_x g_p h_x - f_x g_x h_p + g_x h_p f_x - g_x h_x f_p$
$+ h_x f_p g_x - h_x f_x g_p] = 0$

Define now a space of functions, $\mathcal{F}$ where any function $F(x,p) \in \mathcal{F}$ is in the form

$$F(x,p) = \sum_{n=-\infty}^{\infty} u_n(x,t)\, p^n$$

Here $u_n(x,t)$, are functions of x and t for all integer n. On this space of functions we define a Poisson bracket for and $F, G \in \mathcal{F}$

$$\{F, G\} = p^k \left(\frac{\partial F}{\partial p} \frac{\partial G}{\partial x} - \frac{\partial F}{\partial x} \frac{\partial G}{\partial p} \right)$$

for any integer k.

Let us define the Lax operator (a function in this case) as

$$L = p^{N-1} + \sum_{n=-1}^{N-2} u_n(x,t)\, p^n$$

where N is any positive integer and u_n, $n=-1,0,1,\ldots$ are functions of x and t.

The Lax equation is

$$\frac{\partial L}{\partial t_n} = \{A_n, L\}_k$$

where $A_n = \left(L^{n/N-1}\right)_{\geq -k+1}$

where $n = j + \ell(N-1)$ and $j = 1, 2, \dots, N-1$ and $\ell \in \mathbb{N}$. So we have a hierarchy of for each k and $j = 1, \dots, N-1$. Also $n \geq -k+1$ to ensure that

$$(L^{n/N-1})_{\geq -k+1}$$

is not zero and considered the terms containing the powers of p equal or greater than $-k+1$.

Let

$$L = p + u + v/p \qquad \text{where } u(x,t),\ v(x,t) \text{ are dependent variables}$$

$$L^2 = p^2 + u^2 + v^2/p^2 + 2up + 2v + 2\frac{uv}{p}$$

$$(L^2)_{\geq 0} = p^2 + 2up + u^2 + 2v$$

$$(L^2)_{\geq 1} = p^2 + 2up$$

$$(L)_{\geq 0} = p + u$$

$$(L)_{\geq 1} = p$$

The hierarchy that we obtain by using this Lax representation is sometimes called "dispersionless integrable systems"

$$\frac{dL}{dt_n} = \{ (L^n)_{\geq -k+1} , L \}_k .$$

I) $\underline{k = 0}$

$\underline{n = 1}$

$$\frac{dL}{dt_1} = \{ (L)_{\geq 1} , L \}$$

$$\frac{du}{dt_1} + \frac{1}{p}\frac{dv}{dt_1} = \{ p , p + u + v/p \}$$

$$= u_x + \frac{v_x}{p}$$

$$\frac{du}{dt_1} = u , \quad \frac{dv}{dt_1} = v$$

$n = 2$

$$\frac{dL}{dt_2} = \{ (L^2)_{\geq 1} , L \}$$

$$\frac{du}{dt_2} + \frac{1}{p}\frac{dv}{dt_2} = \{ p^2 + 2up , p + u + \frac{v}{p} \}$$

$$= (2p + 2u)(u_x + \frac{v_x}{p}) - 2u_x p (1 - \frac{v}{p^2})$$

$$= \cancel{2p u_x} + 2v_x + 2u u_x + \frac{2u v_x}{p} - \cancel{2u_x p} + 2u_x v/p$$

$$= 2(v_x + u u_x) + \frac{1}{p}(2u v_x + 2v u_x)$$

$$\frac{du}{dt_2} = 2(v_x + uu_x), \quad \frac{dv}{dt_2} = 2uv_x + 2vu_x$$

$n = 3, \ldots$

II) $k = 1$

$$\frac{dL}{dt_n} = p\left\{ (L^n)_{\geq 0}, L \right\}.$$

$\underline{n=1}$

$$\frac{dL}{dt_1} = p\left\{ L_{\geq 0}, L \right\} = p\left\{ p+u,\ p+u+v/p \right\}$$

$$= p\left(u_x + \frac{v_x}{p} \right) - p u_x \left(1 - v/p^2 \right)$$

$$= \cancel{p u_x} + v_x - \cancel{p u_x} + v u_x \frac{1}{p}.$$

$$\frac{du}{dt_1} = v_x, \quad \frac{dv}{dt_1} = + v u_x.$$

$\underline{n=2}$

$$\frac{dL}{dt_2} = p\left\{ L^2_{\geq 0}, L \right\} = p\left\{ p^2 + 2up + u^2 + 2v,\ p+u+v/p \right\}$$

$$= p\,(2p+2u)\left(u_x + \frac{v_x}{p} \right) - p\left(2u_x p + 2uu_x + 2v_x \right)\left(1 - v/p^2 \right)$$

$$\frac{dL}{dt_2} = p\left(2p\cancel{u}_x + 2\cancel{v}_x + 2\cancel{u}u_x + \underline{2uv_x/p}\right.$$

$$\left. - p\left(2u_x\cancel{p} + 2u\cancel{u}_x + 2\cancel{v}_x - \underline{2vu_x\frac{1}{p}} - 2uu_x v\frac{1}{p^2} - 2vv_x\frac{1}{p^2}\right)\right)$$

$$= 2(uv_x + vu_x) + (2uu_x v + 2vv_x)\frac{1}{p}$$

$$\frac{du}{dt_2} = 2(uv)_x, \quad \frac{dv}{dt_2} = 2v(uu_x + v_x).$$

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$\underline{k=2}$

$$\frac{dL}{dt_n} = p^2\{(L^n)_{\geq 1}, L\} = p^2\{(L^2)_{\geq -1}, L\}$$

$n=1$

$$\frac{dL}{dt_1} = \cancel{p^2\{L_{\geq 1}, L\} = p^2\{p, p+u+v/p\}}$$

$$\cancel{= p^2\left(u_x + \frac{v_x}{p}\right)}$$

$$= p^2\{(L)_{\geq -1}, L\} = p^2\{L, L\} = 0$$

$$\frac{du}{dt_1} = 0, \quad \frac{dv}{dt_1} = 0$$

$$(L^2)_{\geq -1} = p^2 + u^2 + 2v + 2up + \frac{2uv}{p}$$

$n = 2$

$$\frac{dL}{dt_2} = p^2 \left\{ p^2 + u^2 + 2v + 2up + \frac{2uv}{p} \,,\, p + u + \frac{v}{p} \right\}$$

$$= p^2 (2p + 2u)\left(u_x + \frac{v_x}{p}\right) - p^2 \left(2uu_x + 2v_x + 2u_x p + \frac{(2uv)_x}{p}\right)\left(1 - \frac{v}{p^2}\right)$$

$$= p^2 \left[\cancel{2pu_x} + \cancel{2v_x} + \cancel{2uu_x} + \underline{2uv_x} \frac{1}{p} \right]$$

$$- p^2 \left(\cancel{2uu_x} + \cancel{2v_x} + \cancel{2u_x p} + \underline{\frac{(2uv)_x}{p}} \right)$$

$$+ \; v \left(2uu_x + 2v_x + \underline{2u_x p} + \frac{(2uv)_x}{p} \right)$$

$$\frac{du}{dt_2} = 2v(uu_x + v_x) \,, \quad \frac{dv}{dt_2} = 2(uv)_x$$

$n = 3 \ldots$