MATH 337: Introduction to Soliton Theory

Spring 2012 (Lecture 4)

Further properties of the KdV equation

- Hirota's method: The bilinear form
- Backlund Transformations.
- Nonlinear Superposition Rule

Hirota's method : The bilinear form of the KdV equation.

Let

$$u(x,t) = -\frac{\partial^2}{\partial x^2} \ln f$$

where f is a function of (x,t).. First let $u(x,t) = w_x$ then

$$u_t - 6uu_x + u_{xxx} = (w_{tx} - 6 w_x w_{xx} + w_{xxxx}) = 0$$

or

$$(w_t - 3w_x^2 + w_{xxx})_x = 0$$

For rapidly decaying solutions

$$w_t - 3w_x^2 + w_{xxx} = 0$$

let now $\quad w = -2\frac{\partial}{\partial x} \ln f = -2\frac{f_x}{f}$

$$-2\frac{ff_{xt} - f_x f_t}{f^2} - 12\frac{(ff_{xx} - f_x)^2}{f^4} - 2\left(\frac{ff_{xx} - f_x^2}{f^2}\right)_{xx} = 0$$

$$\frac{f f_{xt} - f_x f_t}{f^2} + 6\,\frac{(f f_{xx} - f_x^2)^2}{f^4} + \left(\frac{f f_{xxx} - f_{xx} f_x}{f^2} - \frac{2 f_x (f f_{xx} - f_x^2)}{f^3}\right)_x = 0$$

$$\frac{f f_{xt} - f_x f_t}{f^2} + 6\,\frac{(f f_{xx} - f_x^2)^2}{f^4} + \left(\frac{f f_{xxx} - 3 f_{xx} f_x}{f^2} + \frac{2 f_x^3}{f^3}\right)_x = 0$$

$$\frac{f f_{xt} - f_x f_t}{f^2} + 6\,\frac{(f f_{xx} - f_x^2)^2}{f^4} + \left(\frac{f f_{xxxx} - 2 f_{xxx} f_x - 3 f_{xx}^2}{f^2} + \frac{6 f_x^2 f_{xx}}{f^3} - \frac{2 f_x}{f^3}\,(f f_{xxx} - 3 f_{xx} f_x) - 6\,\frac{f_x^4}{f^4}\right) = 0$$

$$\frac{f f_{xt} - f_x f_t}{f^2} + 6\,\frac{(f f_{xx} - f_x^2)^2}{f^4} + \frac{f f_{xxxx} - 4 f_{xxx} f_x - 3 f_{xx}^2}{f^2} + \frac{12 f_x^2 f_{xx}}{f^3} - \frac{6 f_x^4}{f^4} = 0$$

$$\boxed{f f_{xt} - f_x f_t + f f_{xxxx} + 3 f_{xx}^2 - 4 f_{xxx} f_x = 0}$$

This the bilinear form of the KdV equation

Hirota bilinear operator D_t, D_x

$$D_t^m D_x^n (a \cdot b) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n a(x,t)\, b(x',t') \Big|_{\substack{x'=x \\ t'=t}}$$

for non-negative integers m and n.

Examples

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right) a(x,t)\, b(x',t')$$

$$= \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)\left(a_x(x,t)\, b(x',t') - a(x,t)\, b_{x'}(x',t')\right)$$

$$= a_{xt}\, b(x',t') - a_t(x,t)\, b_{x'}(x',t')$$
$$- a_x(x,t)\, b_{t'}(x',t') + a(x,t)\, b_{x't'}(x',t')$$

$$\Rightarrow \quad D_t D_x (a \cdot b) = a_{xt}\, b - a_t b_x - a_x b_t + a\, b_{xt}$$

if $a = b$.

$$D_t D_x (a \cdot a) = 2(a_{xt}\, a - a_t a_x)$$

$$\mathcal{D}_x^4 (a\cdot b) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^4 (a(x,t)\, b(x',t'))$$

$$= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^3 \left(a_x(x,t)\, b(x',t') - a(x,t)\, b_{x'}(x',t') \right)$$

$$= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^2 \Big(a_{xx}(x,t)\, b(x',t') - a_x(x,t)\, b_{x'}(x',t')$$
$$- a_x(x,t)\, b_{x'}(x',t') + a(x,t)\, b_{x'x'}(x',t') \Big)$$

$$= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^2 \Big[a_{xx}(x,t)\, b(x',t') - 2a_x(x,t)\, b_{x'}(x',t')$$
$$+ a(x,t)\, b_{x'x'}(x',t') \Big]$$

$$= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right) \Big[a_{xxx}(x,t)\, b(x',t') - 2a_{xx}(x,t)\, b_{x'}(x',t')$$
$$+ a_x(x,t)\, b_{x'x'}(x',t') - a_{xx} b_{x'} + 2a_x b_{x'x'}$$
$$- a\, b_{x'x'x} \Big]$$

$$= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right) \Big[a_{xxx}\, b - 3a_{xx}\, b_{x'} + 3a_x\, b_{x'x'} - a\, b_{x'x'x'} \Big]$$

$$= a_{xxxx}\, b - 3a_{xxx}\, b_{x'} + 3a_{xx}\, b_{x'x'} - a_x\, b_{x'x'x}$$
$$- a_{xxx}\, b_{x'} + 3a_{xx}\, b_{x'x'} - 3a_x\, b_{x'x'x'} + a\, b_{x'x'x'x'}$$

$$D_x^4 = a_{xxxx}\, b + a\, b_{xxxx} - 4a_{xxx}\, b_x - 4a_x\, b_{xxx} + 6a_{xx}\, b_{xx}$$

$$D_x^4 (a\cdot a) = 2\, a\, a_{xxxx} - 8a_{xxx}\, a_x + 6\, a_{xx}^2$$

Hence

$$(D_t D_x + D_x^4)\, a\cdot a = 2\left[a a_{xt} - a_t a_x + a a_{xxxx} - 4 a_{xxx} a_x + 3 a_{xx}^2\right]$$

$\Rightarrow$ for KdV we have

$$(D_t D_x + D_x^4)\, f\cdot f = 0$$

is the equation for the KdV equation

Here D_t, D_x are the Hirota operators.

let $B = P(D_t, D_x)$ operator.

for example $P(D_t, D_x) = D_x D_t + D_x^4$

properties.

i) $B(1 \cdot 1) = 0$.

ii) $B(1 \cdot a) = P(D_t, D_x)(1 \cdot a)$

$$= \sum C_{mn} D_t^m D_x^n \quad (1 \cdot a).$$

$$D_x (1 \cdot a) = -\partial_{x'} a.$$

$$D_x^n (1 \cdot a) = (-1)^n \partial_{x'}^n a.$$

$$D_t^m (D_x^n (1 \cdot a)) = (-1)^n D_t^m (\partial_{x'}^n a)$$

$$= (-1)^{m+n} \partial_{t'}^m \partial_{x'}^n a.$$

$$\Rightarrow \quad B(1 \cdot a) = P(-\partial_t, -\partial_x)\, a.$$

$$B(a \cdot 1) = P(\partial_t, \partial_x)\, a.$$

iii) if $a = e^{\theta}$, $\theta = kx - \omega t$

$$B(1 \cdot a) = P(-\partial_t, -\partial_x) e^{kx-\omega t}$$

$$= \sum c_{mn}\, \omega^m (-k)^n e^{kx-\omega t}$$

$$= P(\omega, -k)\, e^{kx-\omega t}$$

$$B(a \cdot 1) = P(-\omega, k)\, e^{kx-\omega t}$$

iv) $B(e^{\theta_i} \cdot e^{\theta_j})$ $\qquad \theta_i = k_i x - \omega_i t$

$$= \sum{}' c_{mn}\, D_t^m D_x^n\, e^{\theta_i} e^{\theta_j}$$

$$D_x^n\, e^{\theta_i} \cdot e^{\theta_j} = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n e^{\theta_i} \cdot e^{\theta_j}$$

$$= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^{n-1} (k_i - k_j)\, e^{\theta_i} \cdot e^{\theta_j}$$

$$= (k_i - k_j)^n\, e^{\theta_i} e^{\theta_j}$$

$$\Rightarrow \quad B(e^{\theta_i} e^{\theta_j}) = P(\omega_j - \omega_i,\ k_i - k_j)\, e^{\theta_i} e^{\theta_j}$$

$$\Rightarrow \quad B(e^{\theta_i} e^{\theta_i}) = 0 \qquad i = j. \quad (P(0,0) = 0)$$

Properties

1) $P(0,0) = 0$

2) $P(D_t, D_x)(1 \cdot 1) = 0$

3) $P(D_t, D_x)(1 \cdot a) = P(-\partial_t, \partial_x)\, a$

4) $P(D_t, D_x)(a \cdot 1) = P(\partial_t, \partial_x)\, a$

5) $P(D_t, D_x)(e^{\theta_i} \cdot e^{\theta_j})$

$$= P(\omega_j - \omega_i, k_i - k_j)\, e^{\theta_i + \theta_j}$$

6) $P(D_t, D_x)(e^{\theta_i} \cdot e^{\theta_i}) = 0 \quad \forall i = 1,2,3,\dots$

7) $P(D_t, D_x)(e^{\theta_i} \cdot e^{\theta_i + \theta_j}) = P(\omega_j, -k_j)\, e^{2\theta_i + \theta_j}$

8) $P(D_t, D_x)(e^{\theta_i + \theta_j} \cdot e^{\theta_i}) = P(-\omega_j, k_j)\, e^{2\theta_i + \theta_j}$

$$\forall\, i, j = 1, 2, 3, \dots$$

Some other examples:

1) Sawada - Kotera equation

$$u_t + 45u^2 u_x - 15 u_x u_{xx} - 15 u u_{xx} + u_{xxxxx} = 0$$

$$u = -2 \frac{\partial^2}{\partial x^2} \ln f.$$

$$D_x (D_t + D_x^5)(f \cdot f) = 0$$

2) Bousinesq equation

$$u_{tt} - u_{xx} + 3(u^2)_{xx} - u_{xxxx} = 0$$

$$u = -2 \frac{\partial^2}{\partial x^2} \ln f$$

$$(D_t^2 - D_x^2 - D_x^4)(f \cdot f) = 0$$

3) Sine - Gordon equation

$$\phi_{xx} - \phi_{tt} = \sin\phi$$

$$\phi = 4 \arctan(g/f)$$

$$(D_x^2 - D_t^2 - 1)(f \cdot g) = 0$$

$$(D_x^2 - D_t^2)(f \cdot f - g \cdot g) = 0$$

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4) Kadomtsev-Petviashvili (KP) equation

$$(u_t - 6uu_x + u_{xxx})_x + 3u_{yy} = 0$$

$$(D_t D_x + D_x^4 + 3D_y^2)(f \cdot f) = 0$$

$$u = -2 \frac{\partial^2}{\partial x^2} \ln f$$

KdV

$$u_t - 6uu_x + u_{xxx} = 0$$

- transformation $u(x,t) = -2(\ln f)_{xx}$
- Hirota bilinear form $(D_x D_t + D_x^4)\{f.f\} = 0$

mKdV

$$u_t + 6pu^2 u_x + u_{xxx} = 0$$

- transformation $u(x,t) = \dfrac{g_x f - g f_x}{g^2 + f^2}$

$$(f^2+g^2)\left[(D_x^3 + D_t)\, g.f\right] + 3(D_x f.g)\left[D_x^2 (f.f + g.g)\right] = 0$$

- Hirota's bilinear form $(D_x^3 + D_t + 3\lambda D_x)\{g.f\} = 0$

$(D_x^2 + \lambda)\{f.f + g.g\} = 0$ λ depends on x and t.

sine-Gordon (sG)

$$\phi_{xx} - \phi_{tt} = \sin\phi$$

- transformation $\phi = 4\arctan(g/f)$

$$\left[(D_x^2 - D_t^2 - 1)(g.f)\right](f^2 - g^2) - fg\left[(D_x^2 - D_t^2)(f.f - g.g)\right] = 0$$

- Hirota's bilinear form $(D_x^2 - D_t - 1)\{g.f\} = 0$

$(D_x^2 - D_t^2)\{f.f - g.g\} = 0$

Nonlinear Schrödinger equation (NLSE)

$$iu_t + u_{xx} + 2\varepsilon|u|^2 u = 0$$

- transformation $u = g/f$

$$f\left[(iD_t + D_x^2)\, gf\right] - g\left[D_x^2 f.f - \varepsilon 2|g|^2\right] = 0$$

- Hirota's bilinear form $(iD_t + D_x^2)\{g.f\} = 0$

$(D_x^2)\{f.f\} = 2\varepsilon|g|^2$

Hirota's perturbation

$$B(f\cdot f) = D_x(D_t + D_x^3)(f\cdot f) = 0$$

Let $\quad f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \varepsilon^4 f_4 + \dots$

where ε is an arbitrary parameter.

$$f\cdot f = (1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \varepsilon^4 f_4 + \dots)(1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \varepsilon^4 f_4 + \dots)$$

$$= (1\cdot 1) + \varepsilon(1\cdot f_1) + \varepsilon(f_1\cdot 1) + \varepsilon^2((1\cdot f_2) + (f_2\cdot 1) + (f_1\cdot f_1))$$
$$+ \varepsilon^3((1\cdot f_3) + (f_3\cdot 1) + (f_1\cdot f_2) + (f_2\cdot f_1))$$
$$+ \varepsilon^4((1\cdot f_4) + (f_4\cdot 1) + f_2\cdot f_2 + f_1\cdot f_3 + f_3\cdot f_1)$$

$$\Rightarrow \quad B(1\cdot 1) = 0$$
$$B(1\cdot f_1) + B(f_1\cdot 1) = 0$$
$$B(1\cdot f_2) + B(f_2\cdot 1) + B(f_1\cdot f_1) = 0$$
$$B(f_3\cdot 1) + B(1\cdot f_3) + B(f_1\cdot f_2) + B(f_2\cdot f_1) = 0$$
$$B(1\cdot f_4) + B(f_4\cdot f_1) + B(f_3\cdot f_2) + B(f_2\cdot f_3) + B(f_2\cdot f_2) = 0$$

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1. One soliton solution.

$$f_1 = e^{\theta}, \qquad \theta = kx - \omega t + \alpha$$

a) $B(1\cdot f_1) + B(f_1 \cdot 1) = 2P(\partial_t \cdot \partial_x) f_1 = 0$

$$D_x(D_t + D_x^3) f_1 = 0 \Rightarrow$$
$$\omega = k^3$$

b). $B(f_1 \cdot f_1) = 0$

$$2B(1\cdot f_2) = 2P(\partial_t, \partial_x) f_2 = 0.$$

Take $f_2 = f_3 = \cdots = f_n = 0.$

hence
$$f = 1 + \varepsilon e^{kx - \omega t + \alpha}, \quad \omega = k^3$$

(take $\varepsilon = 1$ WLG). is a one soliton solution

2. Two soliton solution.

$$f_1 = e^{\theta_1} + e^{\theta_2}, \qquad \theta_1 = k_1 x - \omega_1 x + \alpha_1$$
$$\theta_2 = k_2 x - \omega_2 x + \alpha_2$$

α_1 and α_2 are constants

a) $\omega_1 = k_1^3\ , \ \omega_2 = k_2^3$

$B(f_1 \cdot 1) + B(1 \cdot f_1) = 0$ identically

b) $B(1 \cdot f_2) + B(f_2 \cdot 1) + B(f_1 \cdot f_1) = 0$

$$2\, B(\partial_t, \partial_x) f_2 + B(\cancel{e^{\theta_1} e^{\theta_1}}^{=0} + e^{\theta_1} e^{\theta_2}$$
$$+ e^{\theta_2} e^{\theta_1} + \cancel{e^{\theta_2} e^{\theta_2}}^{=0}) = 0$$

$$2\, B(\partial_t, \partial_x) f_2 + 2\, B(e^{\theta_1} e^{\theta_2}) = 0$$

$$2\, B(\partial_t, \partial_x) f_2 + \cancel{2((k_1 - k_2)(-\omega_1 + \omega_2) + (}$$
$$+ 2\left[(k_1 - k_2)(-\omega_1 + \omega_2) + (k_1^4 - k_2^4)^4\right] e^{\theta_1} e^{\theta_2} = 0$$

$$\Rightarrow \quad f_2 = A\, e^{\theta_1 + \theta_2} = A\, e^{(k_1 + k_2)x - (\omega_1 + \omega_2)t}$$

$$\Rightarrow 2\left[-(k_1 + k_2)(+\omega_1 + \omega_2) + (k_1 + k_2)^4\right] A$$
$$+ 2\ (k_1 - k_2)(-\omega_1 + \omega_2) + (k_1^4 - k_2^4)^4 = 0$$

$$\left[-(k_1+k_2)(k_1^3+k_2^3)+(k_1+k_2)^4\right]A$$ (12)

$$+(k_1-k_2)(-k_1^3+k_2^3)+(k_1-k_2)^4=0$$

$$(k_1+k_2)^2\left[-k_1^2-k_1k_2-k_2^2+k_1^2+2k_1k_2+k_2^2\right]A$$

$$+(k_1-k_2)^2\left[-k_2^2-k_1k_2-k_1^2+k_1^2+k_2^2-2k_1k_2\right]$$

$$A=\left(\frac{k_1-k_2}{k_1+k_2}\right)^2$$

c).

$$2B(f_3\cdot 1)+B(f_1\cdot f_2)+B(f_2\cdot f_1)=0$$

$$B(f_1\cdot f_2)+B(f_2\cdot f_1)=\cancel{P(-\omega_1+\omega_2)(k_1-k_2)}$$

$$=B\left((e^{\theta_1}+e^{\theta_2})\,e^{\theta_1+\theta_2}\right)$$

$$=\cancel{B(e^{2\theta_1+\theta_2})+B}$$

$$=B\left(e^{\theta_1}\,e^{\theta_1+\theta_2}+e^{\theta_2}\cdot e^{\theta_1+\theta_2}\right)$$

$$=P(k_1-(k_1+k_2),\ \omega_1-(\omega_1+\omega_2))$$

$$+P(k_2-(k_1+k_2),\ \omega_2-(\omega_1+\omega_2))$$

$$=\underbrace{P(-k_2,-\omega_2)}_{0}+\underbrace{P(-k_1,-\omega_1)}_{0}$$

Also

$$P(f_2 - f_1) = 0 \quad \text{identically}$$

$$\Rightarrow \quad P(\partial_t, \partial_x) f_3 = 0$$

d) $2B(\partial_t, \partial_x) f_4 + \cdots$

Take $f_3 = 0$

and we know $B(f_2\, f_2) = 0$

$$\Rightarrow \quad 2B(\partial_t, \partial_x) f_4 = 0 \qquad \text{take } f_4 = 0$$

Two soliton solutions

$$f = 1 + \varepsilon(e^{\theta_1} + e^{\theta_2}) + \varepsilon A e^{\theta_1 + \theta_2}$$

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One soliton solution:

$$f = 1 + e^{kx - k^3 t + \alpha}$$

$$u = -2 \frac{\partial^2}{\partial x^2} \ln f = -2 \frac{\partial}{\partial x} \frac{f_x}{f} = -2 \frac{f_{xx} f - f_x^2}{f^2}$$

$$f_x = k\, e^{kx - k^2 t + \alpha}$$

$$\frac{1}{f^2}\left(f_{xx} f - f_x^2\right) = \left(k^2 e^{kx - k^3 t + \alpha}\left(1 + e^{kx - k^3 t + \alpha}\right) - k^2 e^{2kx - 2k^3 t + 2\alpha}\right)\frac{1}{f^2}$$

$$u = -2 \frac{k^2 e^{kx - k^3 t + \alpha}}{\left(1 + e^{kx - k^3 t + \alpha}\right)^2}$$

$$= -2 \frac{k^2}{\left[e^{\frac{1}{2}(kx - k^3 t + \alpha)} + e^{-\frac{1}{2}(\,)}\right]^2}$$

$$u = -\frac{k^2}{2} \cosh^2 \frac{1}{2}\left(kx - k^3 t + \alpha\right)$$

$$= -\frac{k^2}{2} \cosh^2\left[\frac{k}{2}\left(x - kt\right) + \alpha)\right]$$

1 – Soliton solution.

Backlund Transformations

Consider two uncoupled partial differential equations

$$P(u) = 0 \quad \text{and} \quad Q(v) = 0 \tag{1}$$

where P and Q are two operators, which are in general not linear. Let $R_i = 0$ be a set of relations

$$R_i\,(u, v, u_x, v_x, u_t, v_t, \ldots; x, t) = 0 \tag{2}$$

$(i = 1, 2)$ between the two functions u and v. Then $R_i = 0$ is a Bäcklund transformation if it is integrable for v when $P(u) = 0$ and if the resulting v is a solution of $Q(v) = 0$ and vice versa.

If $P = Q$, so that u and v satisfy the same equation, then $R_i = 0$ is called an auto-Backlund transformation. Backlund transformation is useful if the relations $R_i = 0$ are simpler than the original equations (1).

Examples

1. **Laplace equation:**

$$P(u) = u_{xx} + u_{yy} = 0 \quad , \quad Q(v) = v_{xx} + v_{yy} = 0$$

$$R_1 = u_x - v_y = 0$$
$$R_2 = u_y + v_x = 0$$

Auto-Backlund transformations for the Laplace eqn. These are also known as the Cauchy-Riemann relations

proof:

$$u_x = v_y$$
$$u_y = -v_x$$

i) $u_{xx} = v_{yx}$, $u_{yy} = -v_{xy}$ $\Rightarrow P(u) = u_{xx} + u_{yy} = 0$

ii) $u_{xy} = v_{yy}$, $u_{yx} = -v_{xx}$ $\quad Q(v) = v_{yy} + v_{xx} = 0$

Application: $v = xy$ is a solution of the Laplace equation. Then

$u_x = x$, $u_y = -y$ $\Rightarrow$ $u(x,y) = \frac{1}{2}(x^2 - y^2)$ is another solution.

2. Liouville's equation

$$P(u) = u_{xt} - e^{u} = 0 \qquad \text{"Liouville's eqn"}$$

$$Q(v) = v_{xt} \qquad \text{" The wave eqn"}$$

$$R_1 = u_x + v_x - \sqrt{2}\, e^{\frac{1}{2}(u-v)} = 0$$

$$R_2 = u_t - v_t - \sqrt{2}\, e^{\frac{1}{2}(u+v)} = 0$$

proof:

$$u_x + v_x = \sqrt{2}\, e^{\frac{1}{2}(u-v)}$$

$$u_t - v_t = \sqrt{2}\, e^{\frac{1}{2}(u+v)}$$

$$\Rightarrow \text{i)} \quad u_{xt} + v_{xt} = \frac{1}{\sqrt{2}}\, e^{\frac{1}{2}(u-v)} (u_t - v_t)$$

$$= \frac{1}{\sqrt{2}}\, e^{\frac{1}{2}(u-v)} \cdot \sqrt{2}\, e^{\frac{1}{2}(u+v)}$$

$$u_{xt} + v_{xt} = e^{u}$$

ii)

$$u_{xt} - v_{xt} = e^{u}.$$

Adding we get $\quad u_{xt} = e^{u}.$

Subtracting we get $\quad v_{xt} = 0$

Application: most general solution of the $v_{xt} = 0$ equation is $v(x,t) = f(x) + g(t)$ where f and g are arbitrary functions. Using the Backlund transformations we get

$$\left.\begin{aligned} u_x + f_x &= \sqrt{2}\, e^{u/2}\, e^{-1/2(f+g)} \\ u_t - g_t &= \sqrt{2}\, e^{u/2}\, e^{\frac{1}{2}(f+g)} \end{aligned}\right\} \text{B.T.}$$

let

$$u = -2\ln\rho - f + g.$$

Then BTs reduce to

$$\rho_x = -\frac{1}{\sqrt{2}} e^{-f}, \quad \rho_t = -\frac{1}{\sqrt{2}} e^{g}$$

Hence

$$\rho(x,t) = -\frac{1}{\sqrt{2}} \int^x e^{-f(x')} dx' - \frac{1}{\sqrt{2}} \int^t e^{g(t')} dt'$$

$$\Rightarrow \quad u = -2\ln\rho - f + g \quad \Rightarrow \quad e^{u+f-g} = 1/\rho^2$$

$$= -2\ln\left[-\frac{1}{\sqrt{2}} \int^x e^{-f} dx' - \frac{1}{\sqrt{2}} \int^t e^{g} dt'\right] - f + g$$

solves the Liouville equation

3. Sine-Gordon equation.

$$R(u) = u_{xt} - \sin u \quad , \quad \varphi(v) = v_{xt} - \sin v$$

$$R_1 = \frac{1}{2}(u+v)_x - a\sin\left(\frac{u+v}{2}\right) = 0$$

$$R_2 = \frac{1}{2}(u-v)_x - \frac{1}{a}\sin\left(\frac{u-v}{2}\right) = 0$$

Auto-Backlund transformation

proof:

$$(u+v)_x = 2a\sin\left(\frac{u+v}{2}\right)$$

$$(u-v)_t = \frac{2}{a}\sin\left(\frac{u+v}{2}\right)$$

i)
$$u_{xt} + v_{xt} = 2a\cos\left(\frac{u-v}{2}\right)\cdot\frac{2}{2a}\sin\left(\frac{u+v}{2}\right)$$

$$= \cancel{2}\cos\left(\frac{u-v}{2}\right)\sin\left(\frac{u+v}{2}\right)$$

ii)
$$u_{tx} - v_{tx} = \frac{2}{a}\cos\left(\frac{u+v}{2}\right)\frac{2a}{2}\sin\left(\frac{u-v}{2}\right)$$

$$= \cancel{2}\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u+v}{2}\right)$$

Adding $\quad u_{tx} = \sin u$

subtracting $\quad v_{tx} = \sin v$

auto-backlund transf.

<u>Application</u>: A solution of the S-G equation is $v = 0$ $\Rightarrow$ Hence

$$u_{,x} = 2a \sin\left(\frac{u}{2}\right)$$

$$u_{,t} = \frac{2}{a} \sin\left(\frac{u}{2}\right).$$

The first one gives

$$2ax = \int^{u} \frac{du}{\sin(\frac{1}{2}u)} = 2 \tan\left(\tan(u/4)\right) + f(t)$$

Second one gives

$$\frac{2}{a} t = \int^{u} \frac{du}{\sin(u/2)} = 2 \ln\left(\tan(u/4)\right) + g(x)$$

Choose $\quad f(t) = -\frac{2}{a} t + \alpha, \quad g(x) = -2ax + \beta$

α, β are const.

$$\Rightarrow \quad \tan(u/4) = C e^{ax + t/a}$$

or $\quad u(x,t) = 4 \tan^{-1}\left[C e^{ax + t/a}\right]$

Here C is a constant. This is the solitary wave solution of the S-G equation

Backlund transformations for the KdV equation.

1) The Miura transformation

$$u = v^2 + v_x$$

is a backlund transformation, from modified Korteweg-de Vries equation

$$v_t - 6v^2 v_x + v_{xxx} = 0$$

to the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0$$

This is a restricted BT

$$\text{mKdV}(v) \xrightarrow{M.T} \text{KdV}(u)$$

2) Due to the Galilean invariance of KdV equation the Miura transformation takes the form

$$u = \lambda + v^2 + v_x \quad , \quad \lambda = \text{constant}$$

This Miura transformation takes the solutions of the Gardner eqn

$$v_t - 6(v^2+\lambda)v_x + v_{xxx} = 0 \qquad (1)$$

to the solutions of the KdV equation. Hence

$$v_t - 6(v^2+\lambda)v_x + v_{xxx} = 0$$

$$\Downarrow$$

$$u = \lambda + v^2 + v_x$$

$$\Downarrow$$

$$u_t - 6uu_x + u_{xxx} = 0$$

There is a discrete symmetry of (1) if v is a solution of (1) $-v$ is also a solution. For each solution we have different solutions of KdV eqn

$$u_1 = \lambda + v^2 + v_x \qquad (2)$$
$$u_2 = \lambda + v^2 - v_x$$

$$\Rightarrow \quad \left.\begin{aligned} u_1 - u_2 &= 2\vartheta_x \\ u_1 + u_2 &= 2(\lambda + \vartheta^2) \end{aligned}\right\} \quad (3)$$

It is convenient to introduce w such that

$$u_i = \frac{\partial w_i}{\partial x}, \quad i = 1,2 \qquad (4)$$

which corresponds to the potential KdV equation

$$w_t - 3w_x^2 + w_{xxx} = 0 \qquad (5)$$

Then

$$w_1 - w_2 = 2\vartheta \qquad (6)$$

$$\Rightarrow \quad (w_1 + w_2)_x = 2\lambda + \frac{1}{2}(w_1 - w_2)^2 \qquad (7)$$

w_1 and w_2 both satisfy (5) $\Rightarrow$

$$(w_1 - w_2)_t - 3(w_{1x}^2 - w_{2x}^2) + (w_1 - w_2)_{xxx} = 0 \qquad (8)$$

Hence (7) and (8) together provide a BT (auto) transformation for the potential KdV hence for the KdV eqn. due to (4)

Application: A solution of the eq. (5) is $\omega = 0$ so take $\omega_2 = 0$ and find ω_1 from (7) and (8)

$$\omega_{1x} = 2\lambda + \frac{1}{2}(\omega_1)^2 \qquad (9)$$

$$\omega_{1t} - 3\omega_{1x}^2 + \omega_{1xxx} = 0 \qquad (10)$$

(9) can be integrated

$$\frac{d\omega_1}{2\lambda + \frac{1}{2}\omega_1^2} = dx \qquad (11)$$

as

$$\omega_1(x,t) = -2K \tanh(Kx + f(t)) \qquad (12)$$

where $\lambda = -K^2 < 0$. We find $f(t)$ from (10)

$$\omega_{1x} = 2\lambda + \frac{1}{2}\omega_1^2$$

$$\omega_{1xx} = \omega_1 \omega_{1x} = \omega_1 (2\lambda + \frac{1}{2}\omega_1^2)$$

$$\omega_{1xxx} = 2\lambda\, \omega_{1x} + \frac{3}{2}\omega_1^2 \omega_{1x}$$

$$\Rightarrow \omega_{1t} - 3\omega_{1x}^2 + 2\lambda\, \omega_{1x} + \frac{3}{2}\omega_1^2 \omega_{1x} = 0$$

$$\omega_{1t} - 3(2\lambda + \frac{1}{2}\cancel{\omega_1^2})\omega_{1x} + 2\lambda\, \omega_{1x} + \frac{3}{2}\omega_1^2 \cancel{\omega_{1x}} = 0$$

$$\omega_{1t} - 4\lambda\, \omega_{1x} = 0 \quad \text{or}$$

$$\omega_{1t} + 4K^2 \omega_{1x} = 0 \qquad (13)$$

using (12) in (13) we get

$$f' + 4K^3 = 0 \quad \Rightarrow f = -4K^3 t + \alpha.$$

$$\Rightarrow \quad w_1(x,t) = -2k \tanh\left(k(x-4k^2t) + \alpha\right)$$

On the other hand

$$u(x,t) = \frac{\partial}{\partial x} w_1(x,t) = -2k^2 \operatorname{sech}^2[k(x-4k^2t)+\alpha]$$

which is nothing but the solitary wave solution of the KdV equation.

The section is devoted the nonlinear superposition law for integrable systems.

Non-Linear Superposition Law for Soliton Equations

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Abstract

Bäcklund Transformations of sine-Gordon and KdV Equations are studied. The purpose is to generate solutions from a given one. Recursive applications of the Bäcklund Transformations to any solution of these equations generate a hierarchy of solutions. Every such hierarchy thus admits a non-linear superposition principal. At the end a matrix representation of n-th step solution of KdV is given as a conjecture.

1 Introduction

Supose that we have two partial differential equations, in two independent variables x and t, for the two functions u and v. Two equations are expressed as

$$P(u) = 0 \qquad and \qquad Q(v) = 0 \tag{1}$$

where P and Q are two operators, which are in general non linear. Let $R_i = 0$ be a pair of relations

$$R_i(u, v, u_x, v_x, u_t, v_t, ...; x, t) = 0 \qquad i = 1, 2 \tag{2}$$

between the two functions u and v. Then $R_i = 0$ is a Bäcklund Transformation if it is integrable for v when $P(u) = 0$, and if the resulting v is a solution of $Q(v) = 0$, and vice versa. If $P = Q$, so that u and v satisfy the same equation, then $R_i = 0$ is called an **auto- Bäcklund Transformation**. Of course, this approach is normally useful only if the relations $R_i = 0$ are, in some sense, simpler than the original equations.

Example: A fairly simple example arises in connecting with Liouville's equation which we shall define as

Definition: Let x and t be two independent variables and u be a function of x and t. Then the differential equation

$$u_{xt} = e^u \tag{3}$$

is called **Liouville's Equation.**

First we introduce an auxiliary dependent variable, v, which satisfies

$$v_{xt} = 0 \tag{4}$$

Proposition: The pair of the first order differential equations

$$u_x + v_x = \sqrt{2}\, e^{\frac{(u-v)}{2}} \qquad and \qquad u_t - v_t = \sqrt{2}\, e^{\frac{(u+v)}{2}} \tag{5}$$

constitute a Bäcklund Transformation between Liouville's equation and equation (4)

Proof: Let's cross differentiate (5) to obtain

$$u_{xt} + v_{xt} = \frac{1}{\sqrt{2}}(u_t - v_t)e^{\frac{(u-v)}{2}} = e^u \qquad and \tag{6}$$

$$u_{xt} - v_{xt} = \frac{1}{\sqrt{2}}(u_t + v_t)e^{\frac{(u+v)}{2}} = e^u \tag{7}$$

It is clear that the two equations (6) and (7) imply equations (3) and (4).

Example: Let us find a solution of the Liouville's Equation by using Bäcklund Transformation (5). $v = \phi(x) + \psi(y)$ is a solution of (4).Then first equation in (5) becomes

$$u_x + \phi' = \sqrt{2}\, e^{\frac{1}{2}(u-\phi-\psi)} \Longleftrightarrow e^{\phi}(u + \phi - \psi)_x = \sqrt{2}\, e^{\frac{1}{2}(u+\phi-\psi)} \tag{8}$$

Now we define a new independent variable $X(x) = \int^x e^{-\phi}$. Therefore

$$(u + \phi - \psi)_X = \sqrt{2} e^{\frac{1}{2}(u+\phi-\psi)} \Longleftrightarrow -\sqrt{2} e^{-\frac{1}{2}(u+\phi-\psi)} = X + g(y) \tag{9}$$

where $g(y)$ is an arbitrary function of integration. Similarly second equation in (5) gives

$$-\sqrt{2} e^{-\frac{1}{2}(u+\phi-\psi)} = Y + h(x) \tag{10}$$

where $Y(y) = \int^y e^{\psi}$ Therefore

$$e^{\frac{1}{2}(u+\phi-\psi)} = \frac{-\sqrt{2}}{X + Y}$$

hence

$$u = \psi - \phi + 2\ln(\frac{-\sqrt{2}}{X+Y})$$

is the general solution of equation (3) which Liouville(1853) first found by a different method.

2 The sine-Gordon Equation

Definition: Let ψ be a function of two independent parameters u and v. Then the partial differential equation

$$\psi_{uv} = \sin\psi \tag{11}$$

is called sine-Gordon equation.

Proposition: The following transformation

$$\phi_u = \psi_u + 2\lambda\sin(\frac{\phi+\psi}{2}) \tag{12}$$

$$\phi_v = -\psi_v + 2\lambda^{-1}\sin(\frac{\phi-\psi}{2}) \tag{13}$$

between ψ and ϕ is an auto-Bäcklund Transformation for sine-Gordon equation.

Proof: $(12)_v + (13)_u$ gives $\phi_{uv} = \sin\phi$ similarly $(12)_v - (13)_u$ gives (11) so this is an auto-Bäcklund Transformation for sine-Gordon equation.

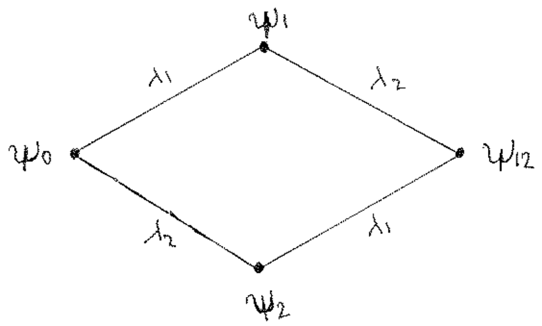
Example: Let us generate a new solution of sine-Gordon equation from the trivial one $\psi_0 = 0$. Putting $\psi_0 = 0$ in (12) and (13), and integrating both equations we get $2\lambda u = 2\ln(\tan\frac{\phi}{4}) + f(v)$ and $2\lambda^{-1}v = 2\ln(\tan\frac{\phi}{4}) + g(u)$. Adding these two equations and then taking exponentials we get

$$\phi = 4\tan^{-1}(c\exp(\lambda u + \frac{v}{\lambda})) \tag{14}$$

where c is arbitrary as $f(v)$ and $g(u)$ are arbitrary. Thus from the trivial solution $\psi_0 = 0$ we generate a new solution (14) which is not that much trivial.

2.1 Nonlinear Superposition Rule for the Sine-Gordon Equation

By a similar way as above one can generate a new solution from ϕ by a new λ. Assume that from ψ_0 we generate ψ_1 and ψ_2 by putting λ_1 and λ_2 into (12) and (13).And assume we generate ψ_{12} from ψ_1 by λ_2, and ψ_{21} from ψ_2 by λ_1. Then Bianchi's Theorem of Permeability states that $\psi_{12} = \psi_{21}$. Using this property we can find ψ_{12} for sine-Gordon Equation.



Proposition: Let ψ_0,ψ_1 and ψ_2 be the solutions of sine-Gordon equation as mentioned above. And let $\psi_3 := \psi_{12} = \psi_{21}$. Then

$$\psi_3 = \psi_0 + 4\tan^{-1}\left(\frac{\lambda_2+\lambda_1}{\lambda_2-\lambda_1}\tan(\frac{1}{4}(\psi_2-\psi_1)\right) \tag{15}$$

Proof: Let ψ_0 be a solution (may be the trivial one)of sine-Gordon equation. Using Bäcklund Transformations we state the followings.

$$\psi_{1u} = \psi_{0u} + 2\lambda_1 \sin(\frac{\psi_0 + \psi_1}{2}) \tag{16}$$

$$\psi_{1v} = -\psi_{0v} + 2\lambda_1^{-1} \sin(\frac{\psi_1 - \psi_0}{2}) \tag{17}$$

$$\psi_{2u} = \psi_{0u} + 2\lambda_2 \sin(\frac{\psi_0 + \psi_2}{2}) \tag{18}$$

$$\psi_{2v} = -\psi_{0v} + 2\lambda_2^{-1} \sin(\frac{\psi_2 - \psi_0}{2}) \tag{19}$$

$$\psi_{3u} = \psi_{1u} + 2\lambda_2 \sin(\frac{\psi_1 + \psi_3}{2}) \tag{20}$$

$$\psi_{3v} = -\psi_{1v} + 2\lambda_2^{-1} \sin(\frac{\psi_3 - \psi_1}{2}) \tag{21}$$

$$\psi_{3u} = \psi_{2u} + 2\lambda_1 \sin(\frac{\psi_2 + \psi_3}{2}) \tag{22}$$

$$\psi_{3v} = -\psi_{2v} + 2\lambda_1^{-1} \sin(\frac{\psi_3 - \psi_2}{2}) \tag{23}$$

(16)-(18) and (20)-(22) together gives

$$\lambda_2 \sin(\frac{\psi_2 + \psi_0}{2}) - \lambda_1 \sin(\frac{\psi_1 + \psi_0}{2}) = \lambda_2 \sin(\frac{\psi_1 + \psi_3}{2}) - \lambda_1 \sin(\frac{\psi_2 + \psi_3}{2}) \tag{24}$$

similarly combining (17)-(19) and (21)-(23) we get

$$\lambda_1 \sin(\frac{\psi_2 - \psi_0}{2}) - \lambda_2 \sin(\frac{\psi_1 - \psi_0}{2}) = \lambda_2 \sin(\frac{\psi_3 - \psi_2}{2}) - \lambda_1 \sin(\frac{\psi_3 - \psi_1}{2}) \tag{25}$$

Using $\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$ on all terms in (24) and (25) and adding side by side we get

$$\frac{\sin\frac{\psi_3}{2} - \sin\frac{\psi_0}{2}}{\cos\frac{\psi_3}{2} + \cos\frac{\psi_0}{2}} = \frac{\lambda_2 + \lambda_1}{\lambda_2 - \lambda_1} \cdot \frac{\sin\frac{\psi_2}{2} - \sin\frac{\psi_1}{2}}{\cos\frac{\psi_2}{2} + \cos\frac{\psi_1}{2}} \tag{26}$$

Now substitute $\frac{\psi_3}{2} = \frac{\psi_3+\psi_0}{4} - \frac{\psi_0-\psi_3}{4}$ and $\frac{\psi_0}{2} = \frac{\psi_3+\psi_0}{4} + \frac{\psi_0-\psi_3}{4}$ and similar expressions for ψ_2 and ψ_1 we get

$$\tan(\frac{\psi_3 - \psi_0}{4}) = \frac{\lambda_2 + \lambda_1}{\lambda_2 - \lambda_1} \tan(\frac{\psi_2 - \psi_1}{4}) \tag{27}$$

Therefore finally we get

$$\psi_3 = \psi_0 + 4\tan^{-1}\left(\frac{\lambda_2 + \lambda_1}{\lambda_2 - \lambda_1} \tan(\frac{\psi_2 - \psi_1}{4})\right) \tag{28}$$

This is the nonlinear superposition rule for the solutions of sine-Gordon equation.

Example: For $\psi_0 = 0$ and under the transformations $u = \frac{1}{2}(x+t)$ and $v = \frac{1}{2}(x-t)$ we can write ψ_1 and ψ_2 from (14) as

$$\psi_1 = 4\tan^{-1}\left[\exp(\frac{x - u_1 t}{\sqrt{1-u_1^2}})\right] \qquad \psi_2 = 4\tan^{-1}\left[\exp(-\frac{x - u_2 t}{\sqrt{1-u_2^2}})\right]$$

where $u_i = \frac{1-\lambda_i^2}{1+\lambda_i^2}$ $\quad i = 1,2$ and so

$$\frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} = \frac{\sqrt{\frac{1-u_1}{1+u_1}} - \sqrt{\frac{1-u_2}{1+u_2}}}{\sqrt{\frac{1-u_1}{1+u_1}} + \sqrt{\frac{1-u_2}{1+u_2}}} = k(u_1, u_2)$$

Then (28) becomes

$$\psi_3 = 4\tan^{-1}\left[k(u_1,u_2)\frac{\exp(\frac{x-u_1t}{\sqrt{1-u_1^2}}) - \exp(-\frac{x-u_2t}{\sqrt{1-u_2^2}})}{1 + \exp(\frac{x-u_1t}{\sqrt{1-u_1^2}})\exp(-\frac{x-u_2t}{\sqrt{1-u_2^2}})}\right] \tag{29}$$

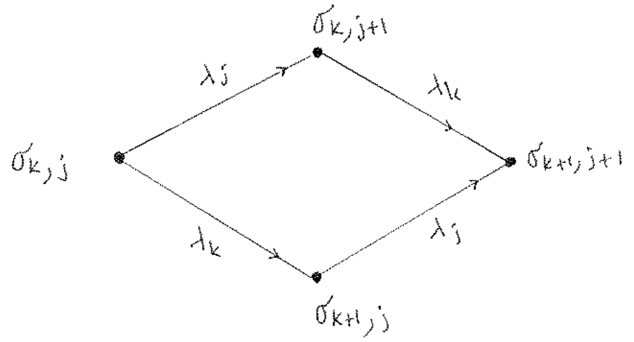
Putting the problem in a centre of mass coordinate system (i.e $u_2 = -u_1$) we obtain

$$\psi_3 = 4\tan^{-1}\left[\frac{u\sinh(\frac{x}{\sqrt{1-u^2}})}{\cosh(\frac{ut}{\sqrt{1-u^2}})}\right] \tag{30}$$

Four solutions of sine-Gordon equation can be related in terms of two constants λ_k and λ_j such that

$$\begin{aligned}
\sigma_{k,j+1} &= \beta_{\lambda_j}\sigma_{k,j} \\
\sigma_{k+1,j} &= \beta_{\lambda_k}\sigma_{k,j} \\
\sigma_{k+1,j+1} &= \beta_{\lambda_j}\beta_{\lambda_k}\sigma_{k,j} = \beta_{\lambda_k}\beta_{\lambda_j}\sigma_{k,j}
\end{aligned} \tag{31}$$

$k, j = 1, 2, \ldots$ where β_{λ_j} is the Bäcklund transformation operator associated with λ_j (equations (12) and (13))



3 The KdV Equation

Definition: Let u be a function of two parameters x and t, then the equation

$$P(u) = u_t - 6uu_x + u_{xxx} = 0 \tag{32}$$

is called the KdV Equation. For Bäcklund Transformation of KdV Equation we will use Wahlquist & Estabrook (1973) approach. Let $u = w_x$ and $Q(w) = w_t - 3w_x^2 + w_{xxx}$. We can easily check that

$$(Q(w))_x = w_{tx} - 6w_x w_{xx} + w_{xxxx} = u_t - 6uu_x + u_{xxx} = P(u)$$

if $P(u) = 0$ then $Q(w) = f(t)$ without lose of generality we may assume $f(t) = 0$. Now let us state Bäcklund Transformation for the KdV Equation.

Proposition: The following two equations

$$w_x + w_x^{'} = 2\lambda + \frac{1}{2}(w - w^{'})^2 \tag{33}$$

$$w_t + w_t^{'} = -(w - w^{'})(w_{xx} - w_{xx}^{'}) + (u^2 + uu^{'} + u^{'2}) \tag{34}$$

give an auto-Bäcklund Transformation for the KdV Equation.

Proof: $(33)_{xx} + (34)$ gives $Q(w) + Q(w^{'}) = 0$, and $(34)_x - (33)_t$ together with $(33)_x$ deduces $Q(w) - Q(w^{'}) = 0$. Therefore w and $w^{'}$ satisfies $Q(w) = Q(w^{'}) = 0$ hence $P(u) = P(u^{'}) = 0$. Therefore (33) and (34) gives an auto-Bäcklund Transformation for the KdV Equation generating a new solution u from a given solution $u^{'}$.

Example: Let $w^{'} = 0$ be the trivial solution of KdV. We can generate a new solution from this trivial one. (33) and (34) becomes

$$w_x = 2\lambda + \frac{1}{2}w^2 \qquad w_t = -ww_{xx} + 2u^2$$

Integrating the first one we get

$$w = -2k\tanh(kx - f(t)) \tag{35}$$

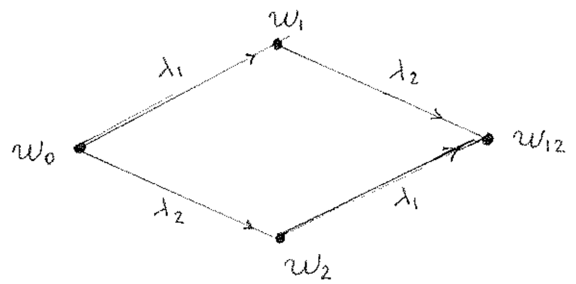
where $\lambda = -k^2$ and $f(t)$ is an integration factor. $f(t)$ can be evaluated from the second equation above, actually $f(t) = -4k^3t + kx_0$. Another singular solution is

$$w = -2k\coth(kx - f(t))$$

.

3.1 Nonlinear Superposition Rule for the KdV Equation

Assume we start with w_0 and produce two new solutions w_1 and w_2 from w_0 by λ_1 and λ_2 respectively. Then taking w_1 and w_2 as original solutions we can produce w_{12} and w_{21} from w_1 and w_2 by λ_2 and λ_1 respectively. Then by the Bianchi's Theorem of Permeability $w_{12} = w_{21}$.



We can easily calculate w_{12} using the Bäcklund transformations for each step.

Proposition:

$$w_{12} = w_0 + 4\frac{\lambda_2 - \lambda_1}{w_2 - w_1} \tag{36}$$

Proof: Writing equation (33) for each step above we get the followings

$$w_{1x} + w_{0x} = -2\lambda_1 + \frac{1}{2}(w_1 - w_0)^2 \tag{37}$$

$$w_{2x} + w_{0x} = -2\lambda_2 + \frac{1}{2}(w_2 - w_0)^2 \tag{38}$$

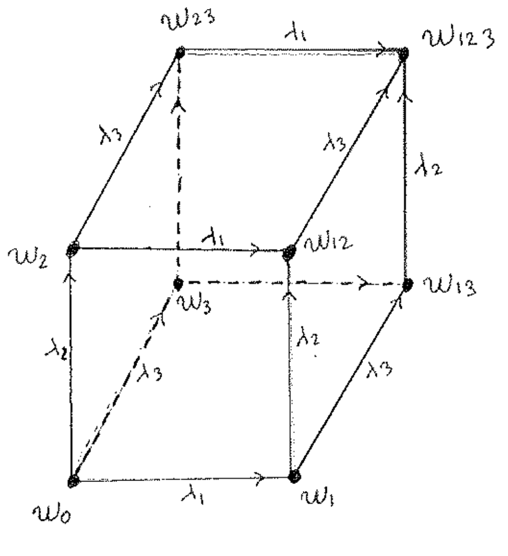
$$w_{12x} + w_{1x} = -2\lambda_2 + \frac{1}{2}(w_{12} - w_1)^2 \tag{39}$$

$$w_{21x} + w_{2x} = -2\lambda_1 + \frac{1}{2}(w_{21} - w_2)^2 \tag{40}$$

Knowing $w_{12} = w_{21}$, (37) - (38) + (40) - (39) gives (36).

3.2 $\mathbf{w_{123}}$ of KdV Equation

We can generate w_{123} from w_{12} by λ_3. As we may change the order of using λ_1, λ_2 and λ_3, we can reach reach w_{123} in several ways as seen in the figure below. Bianchi's Theorem says that in all the ways we get the same solution w_{123}.



Proposition:

$$w_{123} = \frac{\lambda_1 w_1(w_2 - w_3) + \lambda_2 w_2(w_3 - w_1) + \lambda_1 w_1(w_2 - w_3) + \lambda_3 w_3(w_1 - w_2)}{\lambda_1(w_2 - w_3) + \lambda_2(w_3 - w_1) + \lambda_3(w_1 - w_2)} \tag{41}$$

Proof: From (36) we can write the following equations

$$w_{123} = w_1 + 4\frac{\lambda_2 - \lambda_3}{w_{12} - w_{13}} \tag{42}$$

$$w_{12} = w_0 + 4\frac{\lambda_2 - \lambda_1}{w_2 - w_1} \tag{43}$$

$$w_{13} = w_0 + 4\frac{\lambda_3 - \lambda_1}{w_3 - w_1} \tag{44}$$

subtract (44) from (43) and put the result into (42) to get (41).

3.3 n ' th Step Solution of KdV

THEOREM: Let (n) denote the set of n parameters $\{1, 2, ..., n\}$ and $(n)'$ denote the set $\{1, 2, ..., n-1, n+1\}$. Then

$$w_{(n)} = w_{(n-2)} + 4\frac{\lambda_n - \lambda_{n-1}}{w_{(n-1)'} - w_{(n-1)}} \tag{45}$$

Proof: We will use induction. Let $w_{(n-2)}$ be the original solution, and using equation (36) we get the n' th step solution (45).

4 CONJECTURE

4.1 Determinant Representation of $\mathrm{w}_{(\mathrm{n})}$ of the KdV Equation

For even n:

i) n=2: Let

$$U_2 = \begin{vmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{vmatrix}$$

$$V_2 = \begin{vmatrix} 1 & w_1 \\ 1 & w_2 \end{vmatrix}$$

From equation (36) clearly

$$w_{(2)} = \frac{U_2}{V_2}$$

ii) $n = 4$ Let

$$U_4 = \begin{vmatrix} 1 & w_1 & \lambda_1 & \lambda_1^2 \\ 1 & w_2 & \lambda_2 & \lambda_2^2 \\ 1 & w_3 & \lambda_3 & \lambda_3^2 \\ 1 & w_4 & \lambda_4 & \lambda_4^2 \end{vmatrix}$$

$$V_4 = \begin{vmatrix} 1 & w_1 & \lambda_1 & \lambda_1 w_1 \\ 1 & w_2 & \lambda_2 & \lambda_2 w_2 \\ 1 & w_3 & \lambda_3 & \lambda_3 w_3 \\ 1 & w_4 & \lambda_4 & \lambda_4 w_4 \end{vmatrix}$$

After doing some algebra we see that

$$w_4 = \frac{U_4}{V_4}$$

iii) From this two equations I want to make a guess for representation of $w_{(2n)}$. My guess is as follows, which is not proved but it is true for $n = 2, 4$.

$$U_{2n} = \begin{vmatrix} 1 & w_1 & \lambda_1 & \cdots & \lambda_1^{n-2} w_2 & \lambda_1^{n-1} & \lambda_1^n \\ 1 & w_2 & \lambda_2 & \cdots & \lambda_2^{n-2} w_2 & \lambda_2^{n-1} & \lambda_2^n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & w_{2n} & \lambda_{2n} & \cdots & \lambda_{2n}^{n-2} w_{2n} & \lambda_{2n}^{n-1} & \lambda_{2n}^n \end{vmatrix}$$

$$V_{2n} = \begin{vmatrix} 1 & w_1 & \lambda_1 & \cdots & \lambda_1^{n-2} w_2 & \lambda_1^{n-1} & \lambda_1^{n-1} w_1 \\ 1 & w_2 & \lambda_2 & \cdots & \lambda_2^{n-2} w_2 & \lambda_2^{n-1} & \lambda_2^{n-1} w_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & w_{2n} & \lambda_{2n} & \cdots & \lambda_{2n}^{n-2} w_{2n} & \lambda_{2n}^{n-1} & \lambda_{2n}^{n-1} w_{2n} \end{vmatrix}$$

$$w_{2n} = \frac{U_{2n}}{V_{2n}}$$

For odd n:

i) $n = 3$: Let

$$U_3 = \begin{vmatrix} 1 & w_1 & \lambda_1 w_2 \\ 1 & w_2 & \lambda_2 w_2 \\ 1 & w_3 & \lambda_3 w_3 \end{vmatrix}$$

$$V_3 = \begin{vmatrix} 1 & w_1 & \lambda_1 \\ 1 & w_2 & \lambda_2 \\ 1 & w_3 & \lambda_3 \end{vmatrix}$$

then comparing with equation (41) we conclude that

$$w_{(3)} = \frac{U_3}{V_3}$$

ii) For $2n-1$: We will try to make a guess for U_{2n-1} and V_{2n-1}. We will consider two things in our guessing process; firstly part i) , and secondly the equation (45). In equation (45) we see that w_{2n} is obtained from $w_{(2n-1)}$ by reciprocating the difference $w_{(2n-1)'} - w_{(2n-1)}$. So U_{2n-1} should look like V_{2n}

and V_{2n-1} like U_{2n}.Therefore we make our guess as follows:

$$U_{2n-1} = \begin{vmatrix} 1 & w_1 & \lambda_1 & \cdots & \lambda_1^{n-2} & \lambda_1^{n-2} w_1 & \lambda_1^{n-1} w_1 \\ 1 & w_2 & \lambda_2 & \cdots & \lambda_2^{n-2} & \lambda_2^{n-2} w_2 & \lambda_2^{n-1} w_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & w_{2n-1} & \lambda_{2n-1} & \cdots & \lambda_{2n-1}^{n-2} & \lambda_{2n-1}^{n-2} w_{2n-1} & \lambda_{2n-1}^{n-1} w_{2n-1} \end{vmatrix}$$

$$V_{2n-1} = \begin{vmatrix} 1 & w_1 & \lambda_1 & \cdots & \lambda_1^{n-2} & \lambda_1^{n-2} w_1 & \lambda_1^{n-1} \\ 1 & w_2 & \lambda_2 & \cdots & \lambda_2^{n-2} & \lambda_2^{n-2} w_2 & \lambda_2^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & w_{2n-1} & \lambda_{2n-1} & \cdots & \lambda_{2n-1}^{n-2} & \lambda_{2n-1}^{n-2} w_{2n-1} & \lambda_{2n-1}^{n-1} \end{vmatrix}$$

and

$$w_{(2n-1)} = \frac{U_{2n-1}}{V_{2n-1}}$$

4.2 Conclusion

In this paper I studied Bäcklund Transformation for KdV and sine-Gordon equations, and stated Nonlinear Superposition Rule for their solutions. Actually I didn't do anything new. The only new thing I did is the final conjecture for odd n's.

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