MATH 337 Introduction to Soliton Theory

Lecture 3

Spring 2012

Further properties of the Korteweg de Vries equation

1. Conservation laws

2. Lax formulation and KdV hierarchy

3. Hirota's method and bilinear form

4. Backlund Transformations.

1. Conservation Laws.

Let T and X be functions of x and t satisfying

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0 \qquad (1)$$

Eq.(1) is called the "conservation law". We intend to apply this notion to the wave equations, like KdV equation: Hence T and X are functions of $u, u_{x}, u_{xx}, \ldots$ but not t.

Then we expect

$$X \to \text{constant} \quad \text{as} \quad |x| \to \infty$$

Integrating (1) over $(-\infty, \infty)$ we obtain

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} T \, dx + \int_{-\infty}^{\infty} \frac{\partial X}{\partial x} \, dx = 0$$

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} T(x,t) \, dx + \cancel{X \Big|_{-\infty}^{\infty}}^{\to 0} = 0$$

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} T(x,t) \, dx = 0$$

Hence

$$\int_{-\infty}^{\infty} T(x,t)\,dx = \text{constant} \tag{2}$$

The integral of T over all $x \in \mathbb{R}$ is called a "constant of the motion".

For the case of the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{3}$$

i) Let $T = u$ then $X = u_{xx} - 3u^2$

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = u_t - 6uu_x + u_{xxx} = 0$$

Hence we obtain

$$\int_{-\infty}^{\infty} u(x,t)\,dx = \text{constant} \tag{4}$$

This applies for $u, u_x, u_{xx} \to 0$ as $|x| \to \infty$.
(4) does not hold in general for all solutions of the KdV equation. Eq. (4) holds also for the periodic functions (periodic solutions of the KdV equations)

In this case, if the period is p.

$u(x) = u(x+p)$, $u_x(x) = u_x(x+p)$, ...

Then.

$$\int_0^p u(x,t)\,dx = \text{constant}. \qquad (5)$$

ii) If we multiply the KdV equation we get

$$u u_t - 6u^2 u_x + u u_{xxx} = 0$$

$$\frac{1}{2}(u^2)_t - 2(u^3)_x + (u u_{xx})_x - \frac{1}{2}(u_x^2)_x = 0$$

$$T = \frac{1}{2}u^2, \qquad X = -2u^3 + u u_{xx} - \frac{1}{2}u_x^2 \qquad (6)$$

Hence

$$\int_{-\infty}^{\infty} u^2\,dx = \text{constant} \qquad (7)$$

for all solutions of KdV equation which vanish fast enough at $\pm\infty$. For periodic solutions we have

$$\int_0^p u^2\,dx = \text{constant} \qquad (8)$$

In general T is called conserved density for the motion. For the KdV equation u and u^2 are conserved densities of the motion. Physically 4 and 7 imply conservation of mass and momentum during the motion of water in a given region D. There are more conservation laws for the KdV..

iii) $T_3 = u^3 + \frac{1}{2}u_x^2$, $X_3 = -\frac{9}{4}u^4 + 3u^2 u_{xx} - 6u^2 u_x + u_x u_{xxx} - \frac{1}{2}u_{xx}^2$

(show this).

$$\int_{-\infty}^{\infty} (u^3 + \tfrac{1}{2}u_x^2)\,dx = \text{constant} \qquad (9)$$

for all solutions of KdV equation with $u, u_x, u_{xx} \to 0$ as $|x| \to \infty$ and

$$\int_{-\infty}^{\infty} (u^3 + \tfrac{1}{2}u_x^2)\,dx = \text{constant} \qquad (10)$$

also for periodic solutions of the KdV equation. It is, in practice, very difficult to find all the other conservation laws. For instance

iv) $T_4 = 5u^4 + 10uu_x^3 + u_{xx}^2$

$X_4 = \cdots$

$T_5 = 21u^5 + 105u^2u_x^2 + 21uu_{xx}^2 + u_{xxx}^2$

Indeed there are infinitely many conservation laws for the KdV equations, i.e

$T_n, \quad n = 1, 2, 3, \ldots$

$X_n, \quad n = 1, 2, 3, \ldots$

$$\int_{-\infty}^{\infty} T_n(x,t)\, dx = C_n \quad \text{(constant)} \quad n = 1, 2, \ldots \qquad (11)$$

- An infinity of conservation laws

we recall that the Miura transformation

$$u = v^2 + v_x \qquad (12)$$

which led to the connection between the KdV and mKdV equations. Instead of v we work with w defined by

$$v = \frac{1}{2}\varepsilon^{-1} + \varepsilon w \qquad (13)$$

$$\Rightarrow \quad u = \frac{1}{4}\varepsilon^{-2} + w + \varepsilon^2 w^2 + \varepsilon w_x$$

we can omit the first term in the RHS, because of the Galilean invariance of the KdV equation. Hence

$$u = w + \varepsilon^2 w^2 + \varepsilon w_x \qquad (14)$$

This is called the "Gardner transformation"

$$u_t - 6uu_x + u_{xxx} = (1 + \varepsilon\partial_x + 2\varepsilon^2 w)[w_t - 6(w + \varepsilon^2 w^2)w_x + w_{xxx}]$$
$$= 0$$

The Gardner equation

$$w_t - 6(w + \varepsilon^2 w^2)w_x + w_{xxx} = 0 \qquad (15)$$

implied the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0 \qquad (16)$$

A conservation law for the KdV equation is

$$T = w, \quad X = -3w^2 - 2\varepsilon^2 w^3 + w_{xx}$$

$$\int_{-\infty}^{\infty} w(x,t)\,dx = 0 \qquad (17)$$

for all solutions of the Gardner equation with $w, w_x, w_{xx}, \ldots \to 0$ as $|x| \to \infty$.

Using the Gardner transformation and expanding $w(x,t,\varepsilon)$ as a power series of ε.

$$w(x,t) = \sum_{n=0} \varepsilon^n w_n(x,t) = w_0 + \varepsilon w_1 + \cdots$$

we get

$$u = (w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots) + \varepsilon^2 (w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots)^2$$
$$+ \varepsilon (w_{0x} + \varepsilon w_{1x} + \varepsilon^2 w_{2x} + \cdots)$$

$$= w_0 + (w_1 + w_{0x})\varepsilon + (w_2 + w_0^2 + w_{1x})\varepsilon^2$$

~~$+\varepsilon(w_3 + w_1 + w_{2xx})$~~ $+ (w_3 + 2w_0 w_1 + w_{2x})\varepsilon^3$

$+ \cdots$

$w_0 = u$

$w_1 = -w_{0x} = -u_x$

$w_2 = -w_0^2 - w_{1x} = -u^2 + u_{xx}$

$w_3 = +2uu_x - (-u^2 + u_{xx})_x = (2u^2 - u_{xx})_x$

$w_4 = -[2uu_x - (u_{xx} - u^2)_x]_x - 2u(u_{xx} - u^3) - u_x^2$

$\sim +2u^3 + u_x^2$

one can find all w_n's. We have several comments.

i) all odd number w_n's are trivial, that is

$$w_{2n+1} = \Omega_{,x} \quad \text{for some } \Omega$$

with $\Omega \to 0$ as $|x| \to \infty$. Then

$$\int_{-\infty}^{\infty} w_{2n+1}\, dx = 0 \quad \text{identically.}$$

there are trivial conservation laws.

ii) all w_{2n}, $n = 1, 2, \ldots$ correspond to our previous conservation laws.

2) Lax Formulation of the KdV equation and its Hierarchy.

i) Suppose that we wish to solve the initial value problem for u, where $u(x,t)$ satisfies some nonlinear evolution equation of the form

$$u_t = N(u). \tag{18}$$

with $u(x,0) = f(x)$. Here $N(u)$ is symbolic notation for a function which depends on $u, u_x, u_{xx}, u_{xxx}, \ldots$. It may depend also on x. For the KdV equation $N(u) = 6uu_x - u_{xxx}$. We can consider a function space Y so that $u \in Y$.

ii) Let L and M be operators acting on Y so that Eq. (1) is identical with.

$$L_t = [M, L] = ML - LM \tag{19}$$

where L and M are some linear operators in x acting on Y. They may depend on u and its x-derivatives. For the KdV equation

$$L = -D_x^2 + u(x,t), \quad L_t = u_t \tag{20}$$

iii) let Y be a Hilbert space H with inner product (the standard Hilbert space inner product). In general we assume that L is self-adjoint $L = L^{\dagger}$ so that

$$(L\phi, \psi) = (\phi, L^{\dagger}\psi) \qquad (21)$$

for all $\phi, \psi \in H$. We introduce the eigenvalue equation

$$L\psi = \lambda\psi \qquad (22)$$

for $t \geq 0$ and $x \in \mathbb{R}$. where $\lambda = \lambda(t)$ in general. Take the t-derivative of (22).

$$L_t\psi + L\psi_t = \lambda_t\psi + \lambda\psi_t$$

$$[M; L]\psi + (L - \lambda)\psi_t = \lambda_t\psi$$

$$ML\psi - LM\psi + (L - \lambda)\psi_t = \lambda_t\psi$$

$$\lambda M\psi - LM\psi + (L - \lambda)\psi_t = \lambda_t\psi$$

$$(L - \lambda)(\psi_t - M\psi) = \lambda_t\psi \qquad (23)$$

Take inner product with ψ we get

$$(\psi, (L-\lambda)(\psi_t - M\psi)) = \lambda_t (\psi, \psi)$$

$$= ((L-\lambda)\psi, \psi_t - M\psi) = \lambda_t (\psi, \psi)$$

Since $(L-\lambda)\psi = 0 \Rightarrow \lambda_t = 0$. From (23) we get.

$$\psi_t - M\psi = \alpha \psi, \quad \alpha = \text{constant}.$$

Since (19) is invariant under $M \to M + \alpha I$ we can let $\alpha = 0$. Then

$$\psi_t = M\psi \qquad (24)$$

- For the KdV equation:

$$L = -D_x^2 + u = -\frac{\partial^2}{\partial x^2} + u \qquad (25)$$

$$M = -u_x + 2(u + 2\lambda) D_x$$

iv) KdV Hierarchy: Let L be the same operator in the case of KdV, $L = -D_x^2 + u$. but

$$M = A + B \partial_x \qquad (26)$$

let $2A_x + B_{xx} = 0$ or

$$A = -\frac{1}{2} B_x + h(t).$$

$\Rightarrow$

$$u_t = B u_x - \frac{1}{2} B_{xxx} + 2 B_x u - 2\lambda B_x \qquad (29)$$

v) let

$$B = \sum_{n=0}^{N}{}' \, b_{N-n} \lambda^n = b_0 \lambda^N + b_1 \lambda^{N-1} + \cdots + b_N$$

$$u_t = u_x \sum_{n=0}^{N} b_{N-n} \lambda^n - \frac{1}{2} \sum_{n=0}^{N} b_{N-n,xxx} \lambda^n + 2u \sum_{n=0}^{N} b_{N-n,x} \lambda^n$$
$$- 2 \sum_{n=0}^{N} b_{n-N,x} \lambda^{n+1} = 0$$

λ^0: $\quad u_t = b_N u_x - \frac{1}{2} b_{N,xxx} + 2u\, b_{N,x} \qquad (30)$

$n > 0$ $\quad 0 = u_x \sum_{n=1}^{N} b_{N-n} \lambda^n - \frac{1}{2} \sum_{n=1}^{N} b_{N-n\,xxx} \lambda^n$

$$+ 2u \sum_{n=1}^{N} b_{N-n,x} \lambda^n - 2 \sum_{n=0}^{N} b_{n-N,x} \lambda^{n+1} = 0$$

$n+1 = m$

we get

$$u_x b_{N-n} - \frac{1}{2} b_{N-n\,xxx} + 2u\, b_{N-n,x} - 2 b_{N-n+1,x} = 0$$

Let $N-n = k = 1,2,\dots,N-1.$

$$b_{k+1,x} = -\frac{1}{4}\left(D_x^3 - 4uD_x - 2u_x\right) b_k$$

$$= -\frac{1}{4}\left(D^2 - 4u - 2u_x D^{-1}\right) b_{k,x} \qquad (31)$$

Define

$$\mathbb{R} = D^2 - 4u - 2u_x D^{-1}$$

$$b_{k+1,x} = -\frac{1}{4} R\, b_{k,x} \qquad k = 1,2,\dots,N-1. \qquad (32)$$

R is called the "recursion operator".

Then (30) becomes

$$u_t = -\frac{1}{2} R\, b_{N,x} \qquad (33)$$

$$b_{k+1,x} = -\frac{1}{4} R\, b_{k,x} \qquad k = 1,2,\dots$$

where A and B are functions of $x, u, u_x, u_{xx}, \ldots$

Hence

$$\psi_{xx} = (u-\lambda)\psi \qquad (27)$$

$$\psi_t = A\psi + B\psi_x \qquad (28)$$

using

$$L_t = [M, L]$$

$$= [A + BD_x, -D_x^2 + u]$$

$$= (A+BD_x)(-D_x^2+u) - (-D_x^2+u)(A+BD_x)$$

$$= -AD_x^2 + Au + B(-D_x^3 + u_x + uD_x)$$

$$+ D_x(A_x + AD_x + B_x D_x + BD_x^2) - uA - uBD_x$$

$$= -AD_x^2 + B(-D_x^3 + u_x) + A_{xx} + 2A_x D + B_{xx} D_x + AD_x^2$$

$$+ 2B_x D_x^2 + BD_x^3$$

$$L_t\psi = [Bu_x + A_{xx} + (2A_x + B_{xx})D]\psi + 2B_x D_x^2 \psi$$

from (27) $D_x^2\psi = (u-\lambda)\psi$.

$$u_t\psi = [Bu_x + A_{xx} + (2A_x + B_{xx})D]\psi + 2B_x(u-\lambda)\psi$$

Let $b_0 = \text{constant}.$

$\Rightarrow$

$$b_1 = \frac{b_0}{2} u$$

$$b_{k+1,x} = -\frac{1}{4} R\, b_{k,x} = \left(-\frac{1}{4}\right)^2 R^2\, b_{k-1,x}$$

$$= \left(-\frac{1}{4}\right)^k R^k\, b_{1,x} = \frac{b_0}{2}\left(-\frac{1}{4}\right)^k R^k u_{,x}.$$

$$u_t = -\frac{1}{2} R\, b_{N,x} = -\frac{b_0}{4}\left(-\frac{1}{4}\right)^{N-1} R^N u_x$$

$$= b_0\left(-\frac{1}{4}\right)^N R^N u_x.$$

Choose

$$b_0\left(-\frac{1}{4}\right)^N = -1.$$

$$u_t + R^N u_x = 0$$

$$b_{k+1,x} = \frac{b_0}{2}\left(-\frac{1}{4}\right)^k R^k u_x$$

$$= -\frac{b_0}{2}\left(-\frac{1}{4}\right)^{k-N} R^k \qquad k = 1, 2, \ldots$$

N = 1

$$u_t + R u_x = u_t - 6uu_x + u_{xxx} = 0$$

N = 2

$$u_t + R^2 u_x = u_t + R(u_{xxx} - 6uu_x) = 0$$

$$u_t + u_5 - 10uu_{xxx} - 20u_x u_{xx} + 30u^2 u_x = 0$$ (fifth order KdV equation)

N = 3

$$u_t + R^3 u_x = 0$$

This way we obtain the hierarchy of KdV equations

$$u_t + R^N u_x = 0 \quad , \quad N = 1, 2, \ldots$$

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They have the same "recursion operator" R. and they have the same eigenvalue equation or L operator, only M operator changes.

Inverse Scattering of KdV hierarchy

N=1 KdV - we studied this case.

for all N, the initial value problem

$$u_t + R^N u_x = 0$$

$$u(x,0) = f(x) \quad , x \in \mathbb{R}.$$

let $u(x,0) = -2\,\mathrm{sech}^2 x$. We have calculated the scattering data $S(0) = \{ K_n, C_n(0), b(k)=0 \}$. Time evolution of the scattering data $S(t)$ can be obtained from the evolution eqn.

$$\psi_t = A\psi + B\psi_x \quad , \quad A = -\tfrac{1}{2}B_x + h(t).$$

and

$$B = b_0 \lambda^N + b_1 \lambda^{N-1} + \dots$$

as $|x| \to \infty$ all $u, u_x, u_{xx}, \dots \to 0$ hence

$$B = b_0 \lambda^N \quad , \quad A = h(t)$$

$$\psi_t = h(t)\,\psi + b_0 \lambda^N \psi_x$$

since $b_0 = -(-4)^N \Rightarrow$

$$\psi_t = h(t)\psi - (-4)^N \lambda^N \psi_x$$

a) $\lambda < 0$ discrete case $\quad \psi \sim \alpha_n e^{-\kappa_n x}$

$h(t) = 0 \Rightarrow$

$$\alpha_n(t) = \alpha_n(0) e^{+(-4)^N \lambda^N \kappa_n t}, \quad \lambda = -\kappa_n^2$$

$$= \alpha_n(0) e^{4^N \kappa_n^{2N+1} t}$$

b) ~~$x \to \infty$~~ $\lambda > 0$ continuum case $\lambda = k^2$

$x \to \infty \quad \psi = e^{-ikx} + b(k) e^{ikx}$

$$\Rightarrow \dot{a} = 0, \quad b(t) = e^{-2ik^{2N+1}(-4)^N t} b(0)$$

For $u_0(x) = -2\,\mathrm{sech}^2 x$ ~~N~~ two soliton

$b(0) = 0 \Rightarrow b(k,t) = 0$

$$F(x,t) = C_1(0)^2 e^{2(+4)^N \kappa_1^{2N+1} t}$$

$\kappa_1 = 1, \quad C_1 = \sqrt{2}$

$$K(x,z;t) + 2e^{2(+4)^N t - (x+z)} + 2\int K(x,y;t) e^{2(+4)^N t - (y+z)} dy = 0$$

$$K(x, z; t) = L(x,t)\, e^{-z}$$

$$L(x,t) = -\frac{2\, e^{2(+4)^N t - x - y}}{1 + e^{2(-4)^N t - 2x}}$$

$$u(x,t) = -2 \operatorname{sech}^2\left((+4)^N t - x\right).$$

$$= -2 \operatorname{sech}^2\left(4^N t - x\right)$$

$N = 1$ solution of the KdV eqn

$N = 2$ solution of the fifth order KdV equation in page 16

MATH 337
INTRODUCTION TO SOLITON THEORY
Homework set 2

March 1, 2012
Due March 13 , 2012

PROBLEMS.

1. Find, if they exists, the eigenvalues and eigenfunctions of

$$\psi'' + (\lambda - U_0)\,\psi = 0$$

where U_0 is any real constant.

2. Find the eigenvalues and eigenfunctions of

$$\psi'' + (\lambda - u(x))\,\psi = 0$$

where

$$u(x) = \begin{cases} U_0, & 0 < x < 1 \\ 0, & x < 0,\ x > 1 \end{cases}$$

where U_0 is a positive constant.

3. Find the eigenvalues and eigenfunctions of

$$\psi'' + (\lambda - u(x))\,\psi = 0$$

where $u(x) = -U_0\,\delta(x) - U_1\,\delta(x-1)$. Here U_0 and U_1 are positive constants, and show that there is only one one discrete eigenfunction if $(U_0 + U_1)/(U_0\,U_1) > 1$.

4. Find the discrete eigenfunctions for $N = 3$, where $u(x) = -N(N+1)\,sech^2 x$.

Solutions of the homework set 2

1. If $\lambda - U_0 \geq 0$ there is continuous spectrum.

$\lambda - U_0 = \mu^2$

$\psi'' + \mu^2 \psi = 0 \qquad \psi = a e^{i\mu x} + b e^{-i\mu x}$

$\lambda - U_0 = -\mu^2 < 0$

$\psi'' - \mu^2 \psi = 0 \qquad \psi = A e^{-\mu x} + B e^{\mu x}$

we let $\qquad \psi = A e^{-\mu x} \quad$ as $x > 0$

$\qquad \psi = A e^{\mu x} \quad$ as $x < 0$

2. i) If $\lambda = k^2 > 0$ and $\ell^2 = U_0 - k^2 > 0$

then $\quad \psi(x) = A e^{\ell x} + B e^{-\ell x} \qquad 0 < x < 1,$

$\quad \psi(x) = e^{-ikx} + b e^{ikx} \qquad x > 1$

$\quad = a e^{-ikx} \qquad x < 1.$

Continuity of ψ, ψ' at $x = 0$

~~$A + B = 1 + b.$~~

~~$A\ell + B\ell = -ik + bik$~~

$$\left.\begin{aligned} A + B &= a \\ \ell A - \ell B &= -ika \end{aligned}\right\} \quad \begin{aligned} a\ell - ika &= 2\ell A \\ a\ell + ika &= 2\ell B \end{aligned}$$

$$A e^{\ell} + B e^{-\ell} = e^{-ik} + b e^{ik}$$

$$A\ell e^{\ell} - B\ell e^{-\ell} = -ik e^{ik} + ibk e^{ik}$$

$$2\ell A e^{\ell} = (\ell - ik) e^{-ik} + b(\ell + ik) e^{ik}$$

$$2\ell B e^{-\ell} = (\ell + ik) e^{-ik} + b(\ell - ik) e^{ik}$$

$$e^{\ell} a (\ell - ik) = (\ell - ik) e^{-ik} + b(\ell + ik) e^{ik}$$

$$e^{-\ell} a (\ell + ik) = (\ell + ik) e^{-ik} + b(\ell - ik) e^{ik}$$

$$e^{2\ell} \frac{\ell - ik}{\ell + ik} = \frac{(\ell - ik) e^{-ik} + b(\ell + ik) e^{ik}}{(\ell + ik) e^{-ik} + b(\ell - ik) e^{ik}}$$

$$e^{2\ell}(\ell^2 + k^2) e^{-ik} + e^{2\ell} (\ell - ik)^2 e^{ik} b$$

$$= (\ell^2 + k^2) e^{-ik} + b(\ell + ik)^2 e^{ik} b$$

$$(e^{2\ell} - 1)(\ell^2 + k^2) e^{-ik} = b\left[e^{2\ell}(\ell + ik)^2 - (\ell - ik)^2\right]$$

$$b = \frac{(e^{2\ell} - 1)(\ell^2 + k^2) e^{-ik}}{e^{2\ell}(\ell + ik)^2 - (\ell - ik)^2}$$

$$a = \ldots$$

ii) If $\lambda = k^2$ and $k^2 - U_0 = m^2 > 0$. Similar to above by letting $l = im$.

iii) If $\lambda = \sqrt{U_0}$ as above $l \to 0$

and $a(k) = e^{-ik}(1 - \frac{1}{2} ik)$.

3. If $\lambda = k^2 > 0$ then

$$a(k) = 2k^{?} e^{-ik} \left[\{ 2k^2 - ik(U_0 + U_1) \} e^{-ik} + iU_0 U_1 \sin k \right]$$

and $b(k) = \left\{ \frac{iU_0}{2k} + \left(1 - \frac{iU_0}{2k}\right) e^{-2ik} \right\} a - e^{-2ik}$

if $\lambda = -K_n^2 < 0$ then

$$(K_n - \tfrac{1}{2} U_0)(K_n - \tfrac{1}{2} U_1) = \tfrac{1}{4} U_0 U_1 e^{-2K_n}.$$

~~Examine slope of $y = f(x) = (x - \frac{1}{2}U_0)(x - \frac{1}{2}U_1)$ and $y = g(x) = \frac{1}{4} U_0 U_1 e^{-2x}$ at $x = 0$~~

proof is given in the next page

3. $\psi'' + \lambda \psi = 0 \qquad x \neq 0,\ x \neq 1$

$\lambda < -k^2 \qquad \psi = \alpha e^{-kx} \quad x > 1$

$\psi = \beta e^{kx} \quad x < 0$

$\psi = \gamma e^{-kx} + \rho e^{kx} \quad 0 < x < 1$

continuity: at $x=0$ and $x=1$

at $x = 0 \qquad \beta = \gamma + \rho$

at $x = 1 \qquad \alpha e^{-k} = \gamma e^{-k} + \rho e^{k}$

<u>Jump condn.</u>

at $x=0 \qquad \psi'(+) - \psi'(-) = -U_0 \psi(0)$

$-\gamma k + \rho k - \beta k = -U_0 \psi(0) = -U_0 \beta$

$\rho - \gamma - \beta = -U_0 \beta / k$

$\gamma + \rho - \beta = 0$

$\rho = -\frac{1}{2} \beta/k\, U_0, \quad \gamma - \beta = \frac{1}{2} \beta/k\, U_0$

<u>Jump cond. at $x=1$</u>

$\psi'(1+) - \psi'(1-) = -U_1 \psi(1)$

~~$k\beta e^{k} + \alpha e^{-k}$~~ $k = -U_1 \psi(1) = -U_1 \alpha e^{-k}$

$-\alpha k e^{-k} + \gamma k e^{-k} - \rho k e^{k} = -U_1 \alpha e^{-k}$

From $x=0$

$$\beta = \gamma + \rho$$

$$-\gamma + \rho - \beta = -u_0 \beta / k$$

$$\gamma + \rho = \beta$$

$$\rho - \gamma = \left(1 - \frac{u_0}{k}\right)\beta$$

$$2\rho = 2\beta - \beta u_0/k \Rightarrow \rho = \beta\left(1 - \frac{u_0}{2k}\right)$$

$$\gamma = \beta\, u_0/2k$$

From $x=1$

$$\alpha e^{-k} = \gamma e^{-k} + \rho e^{k}$$

$$-\alpha e^{-k} + \gamma e^{-k} - \rho e^{k} = -\alpha u_1 e^{-k}/k$$

$$-2\rho e^{k} = -\alpha u_1 e^{-k}/k$$

$$\rho = \frac{\alpha}{2}\frac{u_1}{k} e^{-2k}$$

$$\Rightarrow \alpha = \gamma + \frac{\alpha}{2}\frac{u_1}{k} \Rightarrow \gamma = \alpha\left(1 - \frac{u_1}{2k}\right)$$

(1) $\rho = \frac{\alpha}{2}\frac{u_1}{k} e^{-2k} = \beta\left(1 - \frac{u_0}{2k}\right)$

(2) $\gamma = \alpha\left(1 - \frac{u_1}{2k}\right) = \beta\, u_0/2k$

$$\frac{\frac{u_1}{2k} e^{-2k}}{1 - \frac{u_1}{2k}} = \frac{1 - \frac{u_0}{2k}}{u_0/2k}$$

$$\frac{u_0 u_1}{4k^2} e^{-2k} = \left(1 - \frac{u_0}{2k}\right)\left(1 - \frac{u_1}{2k}\right)$$

$$\left(K - \frac{u_0}{2k}\right)\left(K - \frac{u_1}{2}\right) = \frac{u_0 u_1}{4} e^{-2K}$$

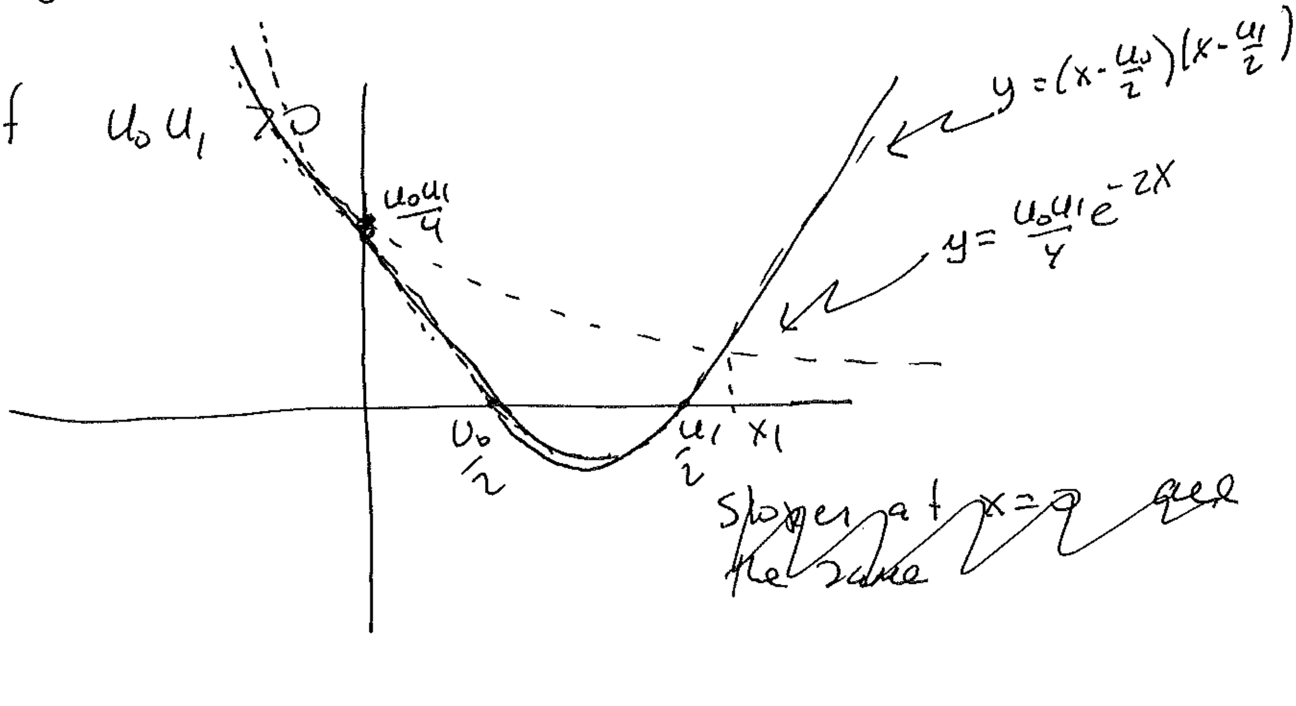
eigen values are found from this eqn

consider the eqn

$$\left(x - \frac{u_0}{2}\right)\left(x - \frac{u_1}{2}\right) = \frac{u_0 u_1}{4} e^{-2x}$$

$$y = \left(x - \frac{u_0}{2}\right)\left(x - \frac{u_1}{2}\right) \quad \text{and} \quad y = \frac{u_0 u_1}{4} e^{-2x}$$

if $u_0 u_1 > 0$



parabola has the slope

$$m_p = y' = 2x - \frac{u_0+u_1}{2} = -\frac{u_0+u_1}{2} \quad \text{at } x=0$$

other curve

$$m_h = y' = -\frac{u_0 u_1}{2} \quad \text{at } x=0$$

if they are the same ✓ or less than 1 $\frac{u_0+u_1}{u_0 u_1} \leq 1$

there exist two eigenvalue $x=0$ and $x=x_1$
the one $x=0$ give zero eigenfunction hence
only one eigenvalue at $x=x_1$

If the slope of parabola is ~~smaller~~ larger

$$\frac{m_p}{m_h} > 1 \Rightarrow \frac{u_0+u_1}{u_0 u_1} > 1$$

parabola intersects the exponential curve
at ~~t~~ $x=0$

The root at $x=0$ is useless, because it
corresponds to zero eigenvalue, but zero
eigenvalue $K_n=0$ corresponds to trivial eigen
function $\psi=0$. We don't consider this situation.
There exist only one eigenvalue K_1 corresponding to
the intersection $x=x_1$.

9. Solution of this problem is straightforward but lengthy. Aşağıdaki kaynakta tüm N-soliton çözümlerini bulabilirsiniz

http://www.ma.utexas.edu/users/uhlen/solitons/notes.pdf

ve ekteki bölümden yararlanılabilir.

Inverse Scattering and Solitons for the KdV Equation

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This term paper deals with the inverse scattering problem and the N-soliton solutions of the KdV equation. It continues the last talk about direct scattering and the initial value problem of the KdV equation. Therefore a lot of references can be found in [2]. For more details and proofs see [1].

The inverse scattering method is used to determine solutions for the KdV equation based on a known initial value q_0 and given scattering data $S_\pm$ at every time. In 1.3 we will see that the scattering matrix S can be reconstructed from both S_- and S_+. The key idea is then to reformulate the scattering relation using Fourier transform to gain the GLM equation. This integral equation can be solved for certain conditions on the scattering data. Finally q is obtained as a partial derivative of this solution.

In the last section N-soliton solutions for the KdV equation are treated. Since these are reflectionless potentials of the Schrödinger equation they are easier to handle and will be solved explicitly.

1 Inverse Scattering

1.1 Premises

As in the direct scattering step we denote by $T_\pm(k) = T(k) = \frac{1}{d(k)}$ the *transmission coefficient*, by $R_\pm(k) = \frac{c_\pm(k)}{d(k)}$ the *reflection coefficients* and by

$$S(k) = \begin{pmatrix} T_-(k) & R_+(k) \\ R_-(k) & T_+(k) \end{pmatrix}$$

the *scattering matrix* $S(k)$. The scattering matrix contains the asymptotic information describing the scattering process. The sets

$$S_\pm = \{R_\pm, k \geq 0; \kappa_j, \gamma_{\pm,j}, j = 1, ..., N\}$$

are called the *scattering data* $S_\pm$ for the Schrödinger operator $H = -\frac{d^2}{dx^2} + q$. For more information on the variables mentioned above see [1] or [2].

1.2 Motivation

Direct scattering starts out with a real valued solution q of the KdV equation

$$q_t - 6qq_x + q_{xxx} = 0$$

and describes how to obtain the scattering data $S_\pm$ for $q_0 = q(.,0)$. It was shown in Theorem 3.1 in [2] that if q satisfies certain conditions then the scattering data $S_\pm(t)$

associated with $q(.,t)$ evolve according to

$$\begin{aligned} S_\pm(t) &= \{\exp(\pm 8ik^3t)R_\pm(k,t=0), k\in\mathbb{R}; \\ &\quad \kappa_j(t=0), \exp(\pm 4\kappa_j^3 t)\gamma_{\pm,j}(t=0), j=1,...,N\}. \end{aligned} \tag{1}$$

The goal in the inverse scattering step is to consider these sets $S_\pm$ (for t fixed) and then find a $q \in L_2^1$ that has scattering data $S_\pm$ and solves the KdV equation.

1.3 Reconstruction of the scattering matrix

First of all we assume that we have already given a scattering matrix and try to reconstruct it using the scattering data:

Theorem 1. *The scattering matrix $S(k)$ can be reconstructed from either of the two sets of scattering data S_+ or S_- and one from the other.*

Sketch of Proof. W.l.o.g. suppose S_+ is given. Define for $\Im k \geq 0$

$$h(k) := T(k)\prod_{i=1}^{N}\frac{k-i\kappa_j}{k+i\kappa_j}.$$

Since $d(k)$ has simple zeros (cf. Theorem 2.4, [2]), $T(k)$ has simple poles at $i\kappa_j$, $j=1,...,N$ and hence h is analytic for $\Im k > 0$ and continuous down to the real line. The only possible zero of h is $k=0$. We therefore distinguish two cases:

1. $d(0)$ is finite, i.e. $h(k)$ has no zeros:
Here $\ln h(k)$ is analytic in the upper half plane and continuous down to the real line. We consider the closed curve Γ_R consisting of the semicircle C_R in the upper half plane and the segment $[-R,R]$ of the real line. Cauchy's integral formula yields

$$\ln h(k) = \frac{1}{2\pi i}\int_{\Gamma_R}\frac{\ln h(\xi)}{\xi-k}d\xi \quad \overset{R\to\infty}{\longrightarrow} \quad \ln h(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln h(\xi)}{\xi-k}d\xi \tag{2}$$

for $\Im k > 0$.

For the function $h(k^*)^*$ and the closed path Γ_R^* in the lower half plane we conclude via Cauchy's theorem that

$$0 = \frac{1}{2\pi i}\int_{\Gamma_R^*}\frac{\ln h(\xi^*)^*}{\xi-k}d\xi \quad \overset{R\to\infty}{\longrightarrow} \quad 0 = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln h(\xi)^*}{\xi-k}d\xi, \tag{3}$$

hence by combining (2) and (3)

$$\ln h(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln|h(\xi)|^2}{\xi-k}d\xi,$$

which finally yields

$$T(k) = \prod_{j=1}^{N}\frac{k+i\kappa_j}{k-i\kappa_j}\exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln(1-|R_+(\xi)|^2)}{\xi-k}d\xi\right\} \tag{4}$$

for $\Im k > 0$, since $|h(\xi)|^2 = |T(\xi)|^2 = 1-|R_\pm(\xi)|^2$.

2. $d(k)$ blows up as $k\to 0$, i.e. $h(0)=0$:
In this case zero is circumvent on a small semicircle c_ε with radius $\varepsilon \leq \frac{1}{2}|k|$ and proved that the contribution of this integral tends to zero as $\varepsilon \to 0$.

All this was done for $\Im k > 0$. For $k \in \mathbb{R}$ we get a slightly different formula using continuity of T:

$$\begin{aligned} T(k) &= \lim_{\epsilon \to 0} T(k + i\epsilon) \\ &= \prod_{j=1}^{N} \frac{k + i\kappa_j}{k - i\kappa_j} \exp\left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln(1 - |R_+(\xi)|^2)}{\xi - k} d\xi + \frac{1}{2} \ln(1 - |R_+(k)|^2) \right\} \quad (5) \end{aligned}$$

where this integral has to be interpreted as Cauchy's principal value.

So far we have obtained $T(k)$ for known $R_+(k)$ and for known values κ_j. R_- is determined by the equation

$$R_+(k)^* T(k) + T(k)^* R_-(k) = 0, \text{[1]} \qquad (6)$$

thus the scattering matrix is reconstructed. Since we know $T(k)$ and $\gamma_{+,j}$ the $\gamma_{-,j}$ can be derived by calculating the residues of T, hence the scattering data S_- is reconstructed as well. □

Note that for Theorem 1 we assumed that the $S_\pm$ are scattering data, i.e. that the $R_\pm$, κ_j and $\gamma_{\pm,j}$ are really what they should be – reflection coefficients, eigenvalues and norming constants. In the inverse scattering step we don't actually have this information and therefore have to find conditions for these variables such that they become scattering data for some q, which needs to be found as well. Theorem 2 states the first requirements to whatever should become S_+.

Theorem 2. *Consider a continuous function $R_+ : \mathbb{R} \to \mathbb{C}$ satisfying the conditions*

1. $R_+(k) = R_+(-k)^*$
2. $|R_+(k)| \leq 1$, $|R_+(k)| = 1 \Rightarrow k = 0$
3. *If* $|R_+(0)| = 1$, *then* $\lim_{k \to 0} \frac{1 + R_+(k)}{k} = \rho_+ \neq 0$
4. $|R_+(k)| = O(\frac{1}{|k|})$ *as* $k \to \infty$

and positive distinct numbers $\kappa_1, ..., \kappa_N$.

Define the function T on $\Im k \geq 0$ by (4) and (5) in the previous proof. Then T is meromorphic in the upper half plane, having only simple poles at $i\kappa_1, ..., i\kappa_N$, and is continuous down to the real axis. It has asymptotic behavior $1 + O(\frac{1}{|k|})$ as $k \to \infty$ and satisfies $T(-k) = T(k)^*$ for real k. Furthermore $|T(k)| > 0$ except possibly if $k=0$. The behavior at $k = 0$ is either $|T(0)| = 0$ or $\lim_{k \to 0} \frac{T(k)}{k} = \alpha \neq 0$.

Next define $R_-(k) := -\frac{R_+(k)^* T(k)}{T(k)^*}$ [2]*. R_- satisfies all the above mentioned conditions on R_+ with ρ_+ replaced by a ρ_-.*

Finally the matrix

$$S(k) = \begin{pmatrix} T(k) & R_+(k) \\ R_-(k) & T(k) \end{pmatrix}, \quad k \in \mathbb{R}$$

is continuous and unitary, in particular $|T(k)|^2 + |R_+(k)|^2 = 1$.

[1]see [1]
[2]cf. (6) above

Proof. The argument of the exponential function in (4) is analytic in $\Im k > 0$ by [3] and the fact that the sequence obtained by integrating only over $[-R, R]$ converges uniformly. Continuity follows by definition. This yields at once the statement on the meromorphicity and continuity of T.
For the rest of the proof see [1]. □

1.4 Gelfand-Levitan-Marchenko equation

The key idea now is to take the Fourier transform of the scattering relation (cf. (2.15), [2])

$$f_\pm(k,x) = c_\mp f_\mp(k,x) + d(k) g_\mp(k,x) \tag{7}$$

and then again use the inverse Fourier transform to gain the Gelfand-Levitan-Marchenko equation (GLM) which will be solved in the next step. In this section we will gain an insight of how to derive the GLM equation from (7).

Recall: The *Fourier transform* of f is denoted by $f^\wedge$, the *inverse Fourier transform* by $f^\vee$ and they are defined by

$$f^\wedge(y) := \int_{-\infty}^{+\infty} f(x) e^{ixy} dx, \qquad f^\vee(x) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x) e^{-ixy} dy.$$

A function h which is analytic in the upper half plane and satisfies

$$\sup_{b>0} \int_{-\infty}^{+\infty} |h(a+ib)|^2 da < \infty$$

is called a *Hardy function*. It can be shown that a function is a Hardy function if and only if it is the Fourier transform of some $f \in L^2(\mathbb{R})$ vanishing on the negative real axis[3]. This also shows that $h|_\mathbb{R}$ is a function in $L^2(\mathbb{R})$.

So how are we going to use this knowledge? Again we suppose that we have already given a $q \in L^1_2(\mathbb{R})$ and the Jost solutions $f_\pm$ of the time-independent Schrödinger equation

$$-f'' + qf = k^2 f.$$

For $\Im k \geq 0$ and $x \in \mathbb{R}$ we introduce the function

$$m_\pm(k,x) := e^{\mp ikx} f_\pm(k,x). \tag{8}$$

It can be shown that $m_\pm(k,x) - 1$ is a Hardy function for each $x \in \mathbb{R}$ and hence is the Fourier transform of a certain function $B_\pm(x,.) \in L^2(\mathbb{R})$ vanishing on the negative real axis.

A similar construction is done for a function $N_\pm$, which depends not only on $f_\pm$ but also on T, κ_j and $\gamma_{\pm,j}$ for $j = 1, ..., N$. $N_\pm$ contains the information about the residues of T. For more details see [1].

Now we are ready to use the (inverse) Fourier transform for (7). For a start we multiply (7) by $e^{\mp ikx}$ on both sides, divide by $d(k)$ and simplify it using $g_\pm(k,x) = f_\pm(-k,x)$ (for k real, cf. (2.13), [2]) to obtain the equation

$$T(k) m_\pm(k,x) = R_\mp(k) e^{\mp 2ikx} m_\mp(k,x) + m_\mp(-k,x). \tag{9}$$

The left hand side may be rewritten in terms of $N_\pm$ and hence be evaluated using the residue theorem. The right hand side yields a product of two Fourier transforms

[3]For a proof see [4].

– of $m_\pm - 1$ and $R_\pm$ (note that we have to assume $R_\pm \in L^2(\mathbb{R})$ so that this makes sense!). Finally we take inverse Fourier transforms on both sides and evaluate at $\mp 2y$.

Thus the following theorem holds:

Theorem 3. *The function $B_\pm(x,y)$ defined by*

$$e^{\mp ikx} f_\pm(k,x) = m_\pm(k,x) = 1 + \int_{-\infty}^{+\infty} B_\pm(x,y) e^{iky} dy \tag{10}$$

for $\Im k \geq 0$ satisfies the Gelfand-Levitan-Marchenko equation

$$B_\pm(x, \pm 2y) + \omega_\pm(x+y) + 2\int_{-\infty}^{+\infty} B_\pm(x, \pm 2z)\omega_\pm(x+y+z)dz = 0 \tag{11}$$

for $\pm y > 0$ where

$$\begin{aligned} \omega_\pm(z) &:= \sum_{j=1}^N \gamma_{\pm,j}^2 e^{\mp 2\kappa_j z} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_\pm(k) e^{\pm 2ikz} dk \qquad (12) \\ &= \sum_{j=1}^N \gamma_{\pm,j}^2 e^{\mp 2\kappa_j z} + R_\pm^\vee(\mp 2z). \qquad \square \end{aligned}$$

Note that the first term of $\omega_\pm$ is derived from the residues of T and the second term from the convolution with $R_\pm$. For a given set of scattering data both ω_+ and ω_- are given functions, since the other set may be calculated from either set by Theorem 1. Hence (11) is an integral equation, which can be used to obtain $B_\pm$ from a given set of scattering data.

If $R_\pm = 0$ we are in the much easier case of the so-called N-soliton solutions (cf. chapter 2).

1.5 Inverse scattering theory

In this section we want to provide more conditions for S_- such that the GLM equation (11) has a unique solution B_- (this applies analogously to B_+). To understand why it is so important to be able to solve the GLM equation we will first establish an actually quite simply connection to q.

Suppose again that $q \in L_2^1(\mathbb{R})$ is given already. Using the definition (10) of B_- via m_- and the unique solution

$$f_\pm(k,x) = e^{\pm ikx} - \int_x^{\pm\infty} \frac{\sin(k(x-x'))}{k} q(x') f_\pm(k,x') dx'$$

of the Volterra integral equation (cf. Theorem 2.1, [2]) one obtains by applying Fourier and inverse Fourier transform that

$$2B_-(x,y) = \int_{-\infty}^{x-y/2} q(x')dx' + \int_0^y \int_{-\infty}^{x+(z-y)/2} q(x') B_-(x',z) dx' dz. \tag{13}$$

A formal proof using iteration yields

Theorem 4. *Let $q \in L_2^1(\mathbb{R})$. Then the integral equation (13) has a (unique) solution $B_-(x,.) : \mathbb{R}_+ \to \mathbb{R}$ such that $B_-(x,.) \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}^+)$ (that implies $B_-(x,.) \in L^2(\mathbb{R})$, so its Fourier transform exists in L^2) and*

$$\frac{d}{dx} B_-(x,0) = \frac{1}{2} q(x) \text{ almost everywhere.} \tag{14}$$

What happens if q isn't given but the scattering data S_- instead? S_- defines the function ω_- and hence the kernel and inhomogeneous term of the GLM equation (11). This equation is a so-called *Fredholm integral equation*. It may be rewritten as

$$A_{-,x}(y) + \underbrace{2\int_0^\infty \omega_{-,x}(y+z)A_{-,x}(z)dz}_{=:F_{-,x}(A_{-,x})(y)} = -\omega_{-,x}(y), \tag{15}$$

where $A_{-,x}(y) := B_-(x, 2y)$ and $\omega_{-,x}(y) := \omega_-(x-y)$. Hence the GLM equation is of the form

$$(1 + F_-)A_- = -\omega_-,$$

where F_- is compact. Since 1 is not an eigenvalue the equation has a unique solution B_- (Fredholm alternative):

Theorem 5. *For distinct positive numbers κ_j and positive numbers $\gamma_{-,j}$, $j = 1, ..., N$, and a function R_- on $\mathbb{R}$ satisfying the conditions of Theorem 2 and having an absolutely continuous inverse Fourier transform $R_-^\vee$ with $(1+|.|)(R_-^\vee)' \in L^1(\mathbb{R})$ the function ω_- defined by (12) is such that the GLM equation (11) has a unique solution B_-.*

By Theorem 4 the potential q can be recovered from

$$q(x) = 2\frac{d}{dx}B_-(x, 0)$$

and we are more or less finished with the inverse scattering step. Two points are still left open:

1. Prove that q is in class L_2^1 and has the set $S_\pm$ as its scattering data.
2. Prove that there is a one-to-one correspondence between q and $S_\pm$.

1.6 Solutions of the KdV equation

Throughout this chapter we always considered t as a fixed parameter. To finally get a solution $q(x,t)$ of the KdV equation two more steps remain to be done:

1. Check that $S_\pm(t)$ evaluated in the time evolution (cf. Theorem 3.1, [2]) forms a set of scattering data, i.e. check that $S_\pm(t)$ satisfies the conditions in Theorem 1 and 5 for each fixed t.
2. Verify that the constructed $q(x,t)$ which was obtained from $q_0(x)$ and the time evolution of the scattering data does indeed satisfy the KdV equation.

2 Solitons

The *N-soliton solution* $q_N(x,t)$ is a special solution of the KdV equation. It is a reflectionless, i.e. $R_\pm = 0$, potential of the Schrödinger equation and has N bound states. Hence it is defined by either of the sets of scattering data

$$S_\pm(t) = \{\kappa_j, \exp(\pm 4\kappa_j^3 t)\gamma_{\pm,j}(0); j = 1, ..., N\}$$

via the inverse scattering method.

The restriction $R_\pm = 0$ simplifies the above equations and the KdV equation becomes solvable via a system of linear equations:

First of all the kernel $\omega_\pm$ of the GLM equation reduces to

$$\omega_\pm(z) = \sum_{j=1}^{N} \gamma_{\pm,j}^2 e^{\mp 2\kappa_j z} = \sum_{j=1}^{N} \gamma_{\pm,j}^2 \omega_{\pm,j}(z),$$

where $\omega_{\pm,j}(z) := e^{\mp 2\kappa_j z}$. Using that $\omega_{\pm,j}(x+y) = \omega_{\pm,j}(x)\omega_{\pm,j}(y)$ and defining $c_{\pm,j}(x) := \gamma_{\pm,j}^2 \omega_{\pm,j}(x)$ the modified GLM equation (15) simplifies to

$$A_{\pm,x}(y) + 2\sum_{j=1}^{N} c_{\pm,j}(x)\omega_{\pm,j}(y)\int_{-\infty}^{+\infty} A_{\pm,x}(z)\omega_{\pm,j}(z)dz = -\sum_{j=1}^{N} c_{\pm,j}(x)\omega_{\pm,j}(y). \quad (16)$$

By making the ansatz $A_{\pm,x}(y) := \sum_{j=1}^{N} \alpha_{\pm,j}(x)\omega_{\pm,j}(y)$ equation (16) yields a system of linear equations for $(\alpha_{\pm,j})_j$. Combining a few results (from the previous chapter and others) one obtains

Theorem 6. *The N-soliton solutions of the KdV equation are of the form*

$$q_N(x,t) = -2\frac{d^2}{dx^2}\ln(\det(1 + \Lambda_N(x,t))),$$

where Λ_N is a $N \times N$ matrix with elements

$$\Lambda_{j,l} = \gamma_{+,j}(0)\gamma_{+,l}(0)\exp(4(\kappa_j^3 + \kappa_l^3)t)(\kappa_j + \kappa_l)^{-1}\exp(-(\kappa_j + \kappa_l)x).$$

Furthermore it can be shown that any N-soliton solution is asymptotically a superposition of N solitary waves of different speed and amplitude as $t \to \pm\infty$. In the general case $R_\pm(k) \neq 0$ the situation is more complicated.

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INVERSE SCATTERING TRANSFORM, KdV, AND SOLITONS

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Abstract: In this review paper, the Korteweg-de Vries equation (KdV) is considered, and it is derived by using the Lax method and the AKNS method. An outline of the inverse scattering problem and of its solution is presented for the associated Schrödinger equation on the line. The inverse scattering transform is described to solve the initial-value problem for the KdV, and the time evolution of the corresponding scattering data is obtained. Soliton solutions to the KdV are derived in several ways.

Mathematics Subject Classification (2000): 35Q53, 35Q51, 81U40
Keywords: Korteweg-de Vries equation, inverse scattering transform, soliton
Short title: KdV and inverse scattering

1. INTRODUCTION

The *Korteweg-de Vries equation* (KdV, for short) is used to model propagation of water waves in long, narrow, and shallow canals. It was first formulated [1] in 1895 by the Dutch mathematicians Diederik Johannes Korteweg and Gustav de Vries. Korteweg was a well known mathematician of his time, and de Vries wrote a doctoral thesis on the subject under Korteweg.

After some scaling, it is customary to write the KdV in the form

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \qquad x \in \mathbf{R}, \quad t > 0, \tag{1.1}$$

where $-u(x,t)$ corresponds to the vertical displacement of the water from the equilibrium at the location x at time t. Replacing u by $-u$ amounts to replacing -6 by $+6$ in (1.1). Also, by scaling x, t, and u, i.e. by multiplying them with some positive constants, it is possible to change the constants in front of each of the three terms on the left-hand side of (1.1) at will.

Note that the KdV is a *nonlinear* partial differential equation (PDE, for short) due to the presence of the uu_x term, where we use a subscript to denote the partial derivative. The u_{xxx} term makes it *dispersive,* i.e. in general an initial wave $u(x,0)$ will broaden in space as time progresses. In addition to its solutions showing behavior of nonlinearity and dispersiveness, the KdV possesses certain special solutions, known as *solitary wave solutions,* which would not be expected from a nonlinear and dispersive PDE. A single solitary wave solution to a PDE retains its shape in time and moves either to the left or right. It turns out that such a solution to (1.1) only moves to the right. In general, any solitary wave solution asymptotically resembles a train of single solitary wave solutions.

It was a Scottish engineer named John Scott Russell who first observed a solitary water wave. This happened in 1834 on the Edinburgh-to-Glasgow canal, some 60 years before the formulation of the KdV as a model for water waves. Russell reported [2] his observation to the British Association in 1844. The details of Russell's observation, including the rivalry between him and George Airy who did not believe in the existence of solitary water waves, can be found in [3-6].

As outlined in the famous preprint [7], which was never published as a journal article, Enrico Fermi in his summer visits to Los Alamos, jointly with J. Pasta and S. Ulam, studied a one-dimensional (1-D, for short) dynamical system of 64 particles with forces between neighbors containing nonlinear terms. These computational studies were carried out on the Los Alamos computer named Maniac I. The primary aim of the study was to determine the rate of approach to the equipartition of energy among various degrees of freedom. Contrary to their expectations from a nonlinear system, Fermi, Pasta, and Ulam observed very little, if any, tendency towards the equipartition of energy, but instead the almost ongoing recurrence to the initial state, which was puzzling. After Fermi's death in November 1954, Pasta and Ulam completed their last few computational examples. Their preprint appears in Fermi's Collected Papers [8] and is also available on the internet [9].

The importance of the KdV arose in 1965, when Zabusky and Kruskal [10] were able to explain the Fermi-Pasta-Ulam puzzle in terms of solitary-wave solutions to the KdV. In their analysis of numerical solutions to the KdV, Zabusky and Kruskal observed *solitary-wave pulses,* named such pulses *solitons* because of their particle-like behavior, and observed that such pulses interact with each other nonlinearly but come out of their interaction virtually unaffected in size or shape. Such unusual nonlinear interactions among soliton solutions to the KdV created a lot of excitement, but at that time no one knew how to solve such a nonlinear PDE, except numerically.

In their celebrated paper [11] of 1967, Gardner, Greene, Kruskal, and Miura presented a method, now known as the *inverse scattering transform,* to solve the initial-value problem for the KdV, assuming that the initial value $u(x, 0)$ approaches a constant sufficiently rapidly as $x \to \pm\infty$. There is no loss of generality in choosing that constant as zero. They showed that $u(x,t)$ can be obtained from $u(x,0)$ with the help of the solution to the *inverse scattering problem* for the 1-D Schrödinger equation with the *time-evolved scattering data.* They also explained that soliton solutions to the KdV corresponded to the case of zero reflection coefficient in the scattering data. They observed from various numerical studies of the KdV that, for large t, $u(x,t)$ in general consists of a finite train of solitons traveling in the positive x direction and an oscillatory train spreading in the opposite direction.

In our paper, we present an elementary review of the inverse scattering transform for the KdV. We consider the time-evolved Schrödinger equation, where $V(x)$ in (2.1) is replaced by $u(x,t)$; namely, we deal with

$$\frac{d^2\psi(k,x;t)}{dx^2}+k^2\,\psi(k,x;t)=u(x,t)\,\psi(k,x;t),\qquad x\in\mathbf{R}, \tag{1.2}$$

where $t>0$ is a parameter that is usually interpreted as time. Thus, we view $V(x)$ as the initial value $u(x,0)$ of the potential $u(x,t)$ and look at $\psi(k,x;t)$ as the time evolution of $\psi(k,x)$ of (2.1) from the initial time $t=0$. The *scattering coefficients* $T(k;t)$, $R(k;t)$, and $L(k;t)$ associated with (1.2) are viewed as evolving from the corresponding coefficients $T(k)$, $R(k)$, and $L(k)$ of (2.1), respectively, from $t=0$. Thus, our notation is such that

$$V(x)=u(x,0),\quad \psi(k,x)=\psi(k,x;0),$$

$$T(k)=T(k;0),\quad R(k)=R(k;0),\quad L(k)=L(k;0).$$

Our review paper is organized as follows. In Section 2 we consider *scattering solutions* and *bound state solutions* to the Schrödinger equation (2.1) and introduce the scattering coefficients, *bound-state norming constants,* and *dependency constants* corresponding to a potential in the so-called *Faddeev class.* In Section 3 we present an outline of the inverse scattering problem for (2.1) and review some solution methods based on solving an associated *Riemann-Hilbert problem.* In Section 4 we consider the time evolution of the scattering coefficients, bound-state norming constants, and dependency constants when the potential evolves from $u(x,0)$ to $u(x,t)$. We also introduce the *Lax pair* associated with the KdV and derive the KdV by using the *Lax method.* In Section 5 we study the *AKNS method* and derive the KdV via that method. In Section 6 we present some of the methods to solve the initial value problem for the KdV. In Section 7 we concentrate on soliton solutions to the KdV and obtain various representations of the N-soliton solution. Finally, in Section 8 we provide certain remarks on the *Bäcklund transformation,* the *conserved quantities,* and some other aspects related to the KdV.

2. SCHRÖDINGER EQUATION AND THE SCATTERING DATA

Consider the Schrödinger equation

$$\frac{d^2\psi(k,x)}{dx^2} + k^2\,\psi(k,x) = V(x)\,\psi(k,x), \qquad x \in \mathbf{R}, \tag{2.1}$$

where V is real valued and belongs to $L^1_1(\mathbf{R})$. Here, $L^1_n(\mathbf{R})$ denotes the class of measurable potentials such that $\int_{-\infty}^{\infty} dx\,(1+|x|^n)\,|V(x)|$ is finite. The class of real-valued potentials in $L^1_1(\mathbf{R})$ is sometimes called the Faddeev class, after Ludwig Faddeev's analysis [12] of the inverse scattering problem for (2.1) within that class of potentials. In appropriate units, (2.1) describes the quantum mechanical behavior of a particle of total energy k^2 under the influence of the potential V. The inverse scattering problem for (2.1) consists of the determination of V from an appropriate set of scattering data.

There are two types of solutions to (2.1). The scattering solutions consist of linear combinations of e^{ikx} and e^{-ikx} as $x \to \pm\infty$, and they occur for $k \in \mathbf{R}\setminus\{0\}$. Real k values correspond to positive energies of the particle, and a particle of positive energy can be visualized as capable of escaping to $\pm\infty$ a result of scattering by V. Heuristically, since $V(x)$ vanishes at $\pm\infty$, the particle will still have some kinetic energy at infinity and hence is allowed to be at infinity. On the other hand, a bound state of (2.1) is a solution that belongs to $L^2(\mathbf{R})$ in the x variable. It turns out that, when V belongs to the Faddeev class, the bound-state solutions to (2.1) decay exponentially as $x \to \pm\infty$, and they can occur only at certain k-values on the imaginary axis in $\mathbf{C}^+$. We use $\mathbf{C}^+$ to denote the upper-half complex plane and $\overline{\mathbf{C}^+} := \mathbf{C}^+ \cup \mathbf{R}$. Each bound state corresponds to a negative total energy of the particle, and as a result the particle is bound by the potential and does not have sufficient kinetic energy to escape to infinity. We will use N to denote the number of bound states, which is known to be finite when V is in the Faddeev class, and suppose that the bound states occur at $k = i\kappa_j$ with the ordering $0 < \kappa_1 < \cdots < \kappa_N$.

Among the scattering solutions to (2.1) are the *Jost solution from the left,* f_{l}, and the *Jost solution from the right,* f_{r}, satisfying the respective boundary conditions

$$e^{-ikx} f_{\mathrm{l}}(k,x) = 1 + o(1), \quad e^{-ikx} f_{\mathrm{l}}'(k,x) = ik + o(1), \qquad x \to +\infty, \tag{2.2}$$

$$e^{ikx} f_{\mathrm{r}}(k,x) = 1 + o(1), \quad e^{ikx} f_{\mathrm{r}}'(k,x) = -ik + o(1), \qquad x \to -\infty, \tag{2.3}$$

where the prime is used for the derivative with respect to the spatial coordinate x. From the spatial asymptotics

$$f_{\mathrm{l}}(k,x) = \frac{e^{ikx}}{T(k)} + \frac{L(k)\, e^{-ikx}}{T(k)} + o(1), \qquad x \to -\infty, \tag{2.4}$$

$$f_{\mathrm{r}}(k,x) = \frac{e^{-ikx}}{T(k)} + \frac{R(k)\, e^{ikx}}{T(k)} + o(1), \qquad x \to +\infty, \tag{2.5}$$

we obtain the scattering coefficients, namely, the *transmission coefficient* T, and the *reflection coefficients* L and R *from the left and right,* respectively. It is also possible to express the scattering coefficients in terms of certain Wronskians [12-16] involving f_{l} and f_{r}. We have

$$T(k) = \frac{2ik}{[f_{\mathrm{r}}(k,x); f_{\mathrm{l}}(k,x)]}, \quad L(k) = \frac{[f_{\mathrm{l}}(k,x); f_{\mathrm{r}}(-k,x)]}{[f_{\mathrm{r}}(k,x); f_{\mathrm{l}}(k,x)]}, \quad R(k) = \frac{[f_{\mathrm{l}}(-k,x); f_{\mathrm{r}}(k,x)]}{[f_{\mathrm{r}}(k,x); f_{\mathrm{l}}(k,x)]}, \tag{2.6}$$

where the Wronskian is defined as $[F;G] := FG' - F'G$.

It is known [12-16] that, for each fixed $x \in \mathbf{R}$, the Jost solutions $f_{\mathrm{l}}(\cdot,x)$ and $f_{\mathrm{r}}(\cdot,x)$ have analytic extensions in k to $\mathbf{C}^+$. Moreover,

$$f_{\mathrm{l}}(-k^*,x) = f_{\mathrm{l}}(k,x)^*, \quad f_{\mathrm{r}}(-k^*,x) = f_{\mathrm{r}}(k,x)^*, \qquad k \in \overline{\mathbf{C}^+},$$

$$T(-k) = T(k)^*, \quad R(-k) = R(k)^*, \quad L(-k) = L(k)^*. \qquad k \in \mathbf{R},$$

where the asterisk denotes complex conjugation. We also have

$$R(k)\,T(k)^* = -L(k)^*\,T(k), \qquad k \in \mathbf{R}, \tag{2.7}$$

$$|T(k)|^2 + |L(k)|^2 = 1 = |T(k)|^2 + |R(k)|^2, \qquad k \in \mathbf{R}. \tag{2.8}$$

Thus, the scattering coefficients cannot exceed one in absolute value for real k. Furthermore, $T(k) \neq 0$ if $k \in \mathbf{R} \setminus \{0\}$, and hence the reflection coefficients are strictly less than one in absolute value when $k \in \mathbf{R} \setminus \{0\}$. In general, R and L are defined only for real k values, but T has a meromorphic extension to $\mathbf{C}^+$. For large k one has

$$T(k) = 1 + O(1/k), \qquad k \to \infty \ \text{ in } \ \overline{\mathbf{C}^+},$$

$$R(k) = o(1/k), \quad L(k) = o(1/k), \qquad k \to \pm\infty.$$

Each bound state corresponds to a pole of T in $\mathbf{C}^+$ and vice versa. It is known that the bound states are simple, i.e. at each $k = i\kappa_j$ there exists only one linearly independent solution to (2.1) belonging to $L^2(\mathbf{R})$. The bound-state norming constants $c_{\mathrm{l}j}$ and $c_{\mathrm{r}j}$ are defined as

$$c_{\mathrm{l}j} := \left[\int_{-\infty}^{\infty} dx\, f_{\mathrm{l}}(i\kappa_j, x)^2\right]^{-1/2}, \quad c_{\mathrm{r}j} := \left[\int_{-\infty}^{\infty} dx\, f_{\mathrm{r}}(i\kappa_j, x)^2\right]^{-1/2},$$

and they are related to each other via the residues of T as

$$\operatorname{Res}(T, i\kappa_j) = i\, c_{\mathrm{l}j}^2\, \gamma_j = i\, \frac{c_{\mathrm{r}j}^2}{\gamma_j}, \tag{2.9}$$

where γ_j is the dependency constant given by

$$\gamma_j := \frac{f_{\mathrm{l}}(i\kappa_j, x)}{f_{\mathrm{r}}(i\kappa_j, x)}. \tag{2.10}$$

The sign of γ_j is the same as that of $(-1)^{N-j}$ and hence

$$c_{\mathrm{r}j} = (-1)^{N-j}\gamma_j c_{\mathrm{l}j}.$$

The normalized bound-state solution $\varphi_j(x)$ at $k = i\kappa_j$ is defined as

$$\varphi_j(x) := c_{\mathrm{l}j}\, f_{\mathrm{l}}(i\kappa_j, x) = (-1)^{N-j} c_{\mathrm{r}j}\, f_{\mathrm{r}}(i\kappa_j, x).$$

The *scattering matrix* associated with (2.1) is given by

$$\mathbf{S}(k) := \begin{bmatrix} T(k) & R(k) \\ L(k) & T(k) \end{bmatrix}, \qquad k \in \mathbf{R},$$

and it can be constructed in terms of the bound-state energies and either one of the reflection coefficients R and L. Given $R(k)$ for $k \in \mathbf{R}$ and the bound-state poles $k = i\kappa_j$, one can construct T as

$$T(k) = \left(\prod_{j=1}^{N} \frac{k + i\kappa_j}{k - i\kappa_j} \right) \exp\left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} ds\, \frac{\log(1 - |R(s)|^2)}{s - k - i0^+} \right), \qquad k \in \overline{\mathbf{C}^+}, \qquad (2.11)$$

and use (2.7) to construct $L(k)$ for $k \in \mathbf{R}$. Similarly, given $L(k)$ for $k \in \mathbf{R}$ and the bound-state poles $k = i\kappa_j$, one can construct T [cf. (2.8) and (2.11)] as

$$T(k) = \left(\prod_{j=1}^{N} \frac{k + i\kappa_j}{k - i\kappa_j} \right) \exp\left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} ds\, \frac{\log(1 - |L(s)|^2)}{s - k - i0^+} \right), \qquad k \in \overline{\mathbf{C}^+},$$

and obtain $R(k)$ for $k \in \mathbf{R}$ via (2.7).

3. INVERSE SCATTERING PROBLEM

When there are no bound states, either one of the reflection coefficients R and L uniquely determines the corresponding potential in the Faddeev class. However, when there are bound states, for the unique determination of V, in addition to one reflection coefficient and the bound-state energies, one must also specify a bound-state norming constant or, equivalently, the dependency constant for each bound state. To recover V uniquely, as our scattering data we may use either the *left scattering data* $\{R, \{\kappa_j\}, \{c_{\mathrm{l}j}\}\}$ or the *right scattering data* $\{L, \{\kappa_j\}, \{c_{\mathrm{r}j}\}\}$; these two are equivalent to each other, and each is also equivalent to $\{\mathbf{S}, \{\gamma_j\}\}$.

A *characterization* for a specific class of potentials consists of specifying some necessary and sufficient conditions on the scattering data which guarantee that there exists a corresponding unique potential in that class. Such conditions are usually obtained by using the *Faddeev-Marchenko method* [12-19]; this method is also known as the *Marchenko method,* and sometimes in the literature it is referred to as the *Gel'fand-Levitan-Marchenko method* even though this is a misnomer [20]. The characterization conditions can be stated for the left scattering data, for the right scattering data, or for the combination of both. For a characterization in the class of real-valued potentials belonging to $L^1_2(\mathbf{R})$, the reader is referred to [14]. Various characterizations in the Faddeev class can be found in [13,17-19].

Since k appears as k^2 in (2.1), the functions $f_\mathrm{l}(-k,x)$ and $f_\mathrm{r}(-k,x)$ are also solutions to (2.1) and they can be expressed as linear combinations of the Jost solutions $f_\mathrm{l}(k,x)$ and $f_\mathrm{r}(k,x)$ as

$$f_\mathrm{l}(-k,x)=T(k)\,f_\mathrm{r}(k,x)-R(k)\,f_\mathrm{l}(k,x),\qquad k\in\mathbf{R},$$

$$f_\mathrm{r}(-k,x)=T(k)\,f_\mathrm{l}(k,x)-L(k)\,f_\mathrm{r}(k,x),\qquad k\in\mathbf{R},$$

or equivalently as

$$m_\mathrm{l}(-k,x)=T(k)\,m_\mathrm{r}(k,x)-R(k)\,e^{2ikx}\,m_\mathrm{l}(k,x),\qquad k\in\mathbf{R},\tag{3.1}$$

$$m_\mathrm{r}(-k,x)=T(k)\,m_\mathrm{l}(k,x)-L(k)\,e^{-2ikx}\,m_\mathrm{r}(k,x),\qquad k\in\mathbf{R},\tag{3.2}$$

where m_l and m_r are the *Faddeev functions* defined as

$$m_\mathrm{l}(k,x):=e^{-ikx}f_\mathrm{l}(k,x),\quad m_\mathrm{r}(k,x):=e^{ikx}f_\mathrm{r}(k,x).\tag{3.3}$$

Each of (3.1) and (3.2) can be viewed as a Riemann-Hilbert problem [6,15], where, knowing the scattering coefficients for $k\in\mathbf{R}$, the aim is to construct m_l and m_r such that, for each $x\in\mathbf{R}$, $m_\mathrm{l}(\cdot,x)$ and $m_\mathrm{r}(\cdot,x)$ are analytic in $\mathbf{C}^+$, continuous in $\overline{\mathbf{C}^+}$, and behave like $1+O(1/k)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$. Once $m_\mathrm{l}(k,x)$ or $m_\mathrm{r}(k,x)$ is constructed, the potential can be obtained with the help of (2.1) and (3.3), namely, by using

$$V(x)=\frac{m_\mathrm{l}''(k,x)}{m_\mathrm{l}(k,x)}+2ik\frac{m_\mathrm{l}'(k,x)}{m_\mathrm{l}(k,x)},\quad V(x)=\frac{m_\mathrm{r}''(k,x)}{m_\mathrm{r}(k,x)}-2ik\frac{m_\mathrm{r}'(k,x)}{m_\mathrm{r}(k,x)},\tag{3.4}$$

where the right-hand sides can be evaluated at any particular value of $k \in \overline{\mathbf{C}^+}$.

Alternatively, the potential can be constructed by the Faddeev-Marchenko method; namely, V can be obtained from the left scattering data $\{R, \{\kappa_j\}, \{c_{\mathrm{l}j}\}\}$ by solving the *left Marchenko integral equation* or from the right scattering data $\{L, \{\kappa_j\}, \{c_{\mathrm{r}j}\}\}$ by solving the *right Marchenko integral equation.*

The left Marchenko equation using the left scattering data as the input is given by

$$B_{\mathrm{l}}(x,y) + \Omega_{\mathrm{l}}(2x+y) + \int_0^\infty dz\, \Omega_{\mathrm{l}}(2x+y+z)\, B_{\mathrm{l}}(x,z) = 0, \qquad y > 0, \tag{3.5}$$

where

$$\Omega_{\mathrm{l}}(y) := \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, R(k)\, e^{iky} + \sum_{j=1}^{N} c_{\mathrm{l}j}^2\, e^{-\kappa_j y}.$$

One can obtain (3.5) from (3.1) via a Fourier transformation. Once (3.5) is solved and $B_{\mathrm{l}}(x,y)$ is obtained, the potential is recovered as

$$V(x) = -2\,\frac{dB_{\mathrm{l}}(x,0^+)}{dx}, \tag{3.6}$$

and the Faddeev function from the left is constructed as

$$m_{\mathrm{l}}(k,x) = 1 + \int_0^\infty dy\, B_{\mathrm{l}}(x,y)\, e^{iky}. \tag{3.7}$$

Similarly, via a Fourier transformation on (3.2), using the right scattering data as the input one obtains the right Marchenko equation

$$B_{\mathrm{r}}(x,y) + \Omega_{\mathrm{r}}(-2x+y) + \int_0^\infty dz\, \Omega_{\mathrm{r}}(-2x+y+z)\, B_{\mathrm{r}}(x,z) = 0, \qquad y > 0, \tag{3.8}$$

where

$$\Omega_{\mathrm{r}}(y) := \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, L(k)\, e^{iky} + \sum_{j=1}^{N} c_{\mathrm{r}j}^2\, e^{-\kappa_j y}.$$

Once (3.8) is solved, the potential is recovered by using

$$V(x) = 2\,\frac{dB_{\mathrm{r}}(x,0^+)}{dx}, \tag{3.9}$$

and the Faddeev function from the right is constructed as

$$m_{\mathrm{r}}(k,x)=1+\int_0^{\infty} dy\, B_{\mathrm{r}}(x,y)\, e^{iky}. \tag{3.10}$$

When the characterization conditions on the scattering data corresponding to potentials in the Faddeev class are satisfied, both the left and right Marchenko equations are uniquely solvable, and the right-hand sides of (3.6) and (3.9) are equal to each other and belong to $L_1^1(\mathbf{R})$. Thus, V can be obtained from either (3.6) or (3.9). There are various other methods to recover V from an appropriate set of scattering data. We refer the reader to [16,19] for a review of some of those methods.

4. LAX METHOD AND EVOLUTION OF THE SCATTERING DATA

Soon after Gardner, Green, Kruskal, and Miura showed [11] that the initial value problem for the KdV can be solved by the inverse scattering transform, Peter Lax gave [21] a criterion to show that the KdV can be viewed as a *compatibility condition* related to the time evolution of solutions to (1.2). Since Lax's criterion is applicable to other nonlinear PDEs solvable by an inverse scattering transform (e.g. the initial-value problem for the nonlinear Schrödinger equation can be solved [4,6,22] via the inverse scattering transform for the Zakharov-Shabat system), we first outline the general idea behind the Lax method and next demonstrate its application on the KdV.

Given a linear operator $\mathcal{L}$ with $\mathcal{L}\psi=\lambda\psi$, we are interested in finding another operator $\mathcal{A}$ (the operators $\mathcal{A}$ and $\mathcal{L}$ are said to form a Lax pair) such that:

(i) The spectral parameter λ does not change in time.

(ii) The quantity $\psi_t-\mathcal{A}\psi$ must remain a solution to $\mathcal{L}\psi=\lambda\psi$.

(iii) The quantity $\mathcal{L}_t+\mathcal{L}\mathcal{A}-\mathcal{A}\mathcal{L}$ must be a multiplication operator.

As the following argument shows. for compatibility, we are forced to have

$$\mathcal{L}_t + \mathcal{L}\mathcal{A} - \mathcal{A}\mathcal{L} = 0, \tag{4.1}$$

which is interpreted as an integrable PDE and in general is nonlinear. From condition (ii) above we see that

$$\mathcal{L}\left(\psi_t - \mathcal{A}\psi\right) = \lambda\left(\psi_t - \mathcal{A}\psi\right),$$

or equivalently,

$$\begin{aligned} \mathcal{L}\psi_t - \mathcal{L}\mathcal{A}\psi &= \lambda\psi_t - \mathcal{A}\left(\lambda\psi\right) \\ &= \partial_t\left(\lambda\psi\right) - \mathcal{A}\mathcal{L}\psi \\ &= \partial_t\left(\mathcal{L}\psi\right) - \mathcal{A}\mathcal{L}\psi \\ &= \mathcal{L}_t\psi + \mathcal{L}\psi_t - \mathcal{A}\mathcal{L}\psi, \end{aligned} \tag{4.2}$$

where we have used $\mathcal{L}\psi = \lambda\psi$ and $\lambda_t = 0$. After canceling the $\mathcal{L}\psi_t$ terms in (4.2), we get

$$\left(\mathcal{L}_t + \mathcal{L}\mathcal{A} - \mathcal{A}\mathcal{L}\right)\psi = 0. \tag{4.3}$$

Because of condition (iii) listed above, from (4.3) we obtain the compatibility condition (4.1).

Let us write the Schrödinger equation (1.2) as

$$\mathcal{L}\psi = \lambda\psi, \quad \mathcal{L} := -\partial_x^2 + u(x,t), \tag{4.4}$$

and try

$$\mathcal{A} = \alpha\partial_x^3 + \beta\partial_x^2 + \xi\partial_x + \eta, \tag{4.5}$$

where the coefficients α, β, ξ, and η may depend on x and t, but not on λ. Note that $\mathcal{L}_t = u_t$ and λ is the same as k^2. Using (4.4) and (4.5) we can write (4.1) explicitly as

$$(\ )\partial_x^5 + (\ )\partial_x^4 + (\ )\partial_x^3 + (\ )\partial_x^2 + (\ )\partial_x + (\ ) = 0, \tag{4.6}$$

where each ( ) in (4.6) denotes the appropriate coefficient. The coefficient of ∂_x^5 automatically vanishes. Setting the coefficients of ∂_x^j equal to zero for $j = 4, 3, 2, 1$, we obtain

$$\alpha = c_1, \quad \beta = c_2, \quad \xi = c_3 - \frac{3}{2} c_1 u, \quad \eta = c_4 - \frac{3}{4} c_1 u_x - c_2 u,$$

where c_1, c_2, c_3, and c_4 are arbitrary constants. Using $c_1 = -4$ and $c_3 = 0$ in the last term on the left-hand side of (4.6) and setting that last term to zero, we obtain the KdV equation given in (1.1). Further, by using $c_2 = c_4 = 0$, we obtain the operator $\mathcal{A}$ associated with $\mathcal{L}$ as

$$\mathcal{A} = -4\partial_x^3 + 6u\partial_x + 3u_x. \tag{4.7}$$

Let us remark that condition (ii) stated above allows us to determine the time evolution of any solution to the Schrödinger equation (1.2) as the initial potential $u(x, 0)$ evolves to $u(x, t)$. For example, let us find the time evolution of $f_{\mathrm{l}}(k, x; t)$, the Jost solution from the left. By using (2.2) and (2.5) with $T(k)$ and $R(k)$ replaced by $T(k; t)$ and $R(k; t)$, respectively, we see that

$$e^{-ikx} f_{\mathrm{l}}(k, x; t) = 1 + o(1), \quad e^{-ikx} f_{\mathrm{l}}'(k, x; t) = ik + o(1), \qquad x \to +\infty, \tag{4.8}$$

$$f_{\mathrm{r}}(k, x; t) = \frac{e^{-ikx}}{T(k; t)} + \frac{R(k; t)\, e^{ikx}}{T(k; t)} + o(1), \qquad x \to +\infty. \tag{4.9}$$

From condition (ii) of the Lax method and (4.7) we obtain

$$\partial_t f_{\mathrm{l}}(k, x; t) - \left(-4\partial_x^3 + 6u\partial_x + 3u_x\right) f_{\mathrm{l}}(k, x; t) = p(k, t)\, f_{\mathrm{l}}(k, x; t) + q(k, t)\, f_{\mathrm{r}}(k, x; t), \tag{4.10}$$

where we have used the fact that the quantity $\partial_t f_{\mathrm{l}} - \mathcal{A} f_{\mathrm{l}}$ remains a solution to (1.2) and hence can be expressed as a linear combination of the two linearly independent Jost solutions $f_{\mathrm{l}}(k, x; t)$ and $f_{\mathrm{r}}(k, x; t)$ with coefficients $p(k, t)$ and $q(k, t)$, respectively. For each fixed t, assuming that $u(x, t) = o(1)$ and $u_x(x, t) = o(1)$ as $x \to +\infty$, the coefficients $p(k, t)$ and $q(k, t)$ can be evaluated by letting $x \to +\infty$ in (4.10). Using (4.8) and (4.9) in (4.10), we get

$$\partial_t e^{ikx} + 4\partial_x^3 e^{ikx} = p(k, t)\, e^{ikx} + q(k, t) \left[\frac{1}{T(k; t)}\, e^{-ikx} + \frac{R(k; t)}{T(k; t)}\, e^{ikx}\right]. \tag{4.11}$$

From (4.11), by comparing the coefficients of e^{ikx} and e^{-ikx} on both sides, we obtain

$$q(k,t) = 0, \quad p(k,t) = -4ik^3. \tag{4.12}$$

Thus, the time evolution of $f_l(k,x;t)$ is determined by the linear third-order PDE

$$\partial_t f_l(k,x;t) - \mathcal{A} f_l(k,x;t) = -4ik^3\, f_l(k,x;t). \tag{4.13}$$

Similarly, letting $x \to -\infty$ in (4.10), with the help of (2.3), (2.4), and (4.12) we obtain

$$\partial_t \left[\frac{1}{T(k;t)}\, e^{ikx} + \frac{L(k;t)}{T(k;t)}\, e^{-ikx}\right] = \left[-4\partial_x^3 - 4ik^3\right] \left[\frac{1}{T(k;t)}\, e^{ikx} + \frac{L(k;t)}{T(k;t)}\, e^{-ikx}\right], \tag{4.14}$$

where we have also used $u(x,t) = o(1)$ and $u_x(x,t) = o(1)$ as $x \to -\infty$. From (4.14), comparing the coefficients of e^{ikx} and e^{-ikx} on both sides, we obtain

$$\partial_t T(k;t) = 0, \quad \partial_t L(k;t) = -8ik^3 L(k;t),$$

and hence

$$T(k;t) = T(k;0) = T(k), \quad L(k;t) = L(k;0)\, e^{-8ik^3 t} = L(k)\, e^{-8ik^3 t}. \tag{4.15}$$

Thus, the transmission coefficient remains unchanged and the reflection coefficient from the left undergoes a simple phase change as t progresses.

Proceeding in a similar manner, we can obtain the time evolution of the Jost solution $f_r(k,x;t)$ and the right reflection coefficient $R(k;t)$. With $\mathcal{A}$ as in (4.7), we get

$$\partial_t f_r(k,x;t) - \mathcal{A} f_r(k,x;t) = 4ik^3 f_r(k,x;t), \tag{4.16}$$

$$R(k;t) = R(k;0)\, e^{8ik^3 t} = R(k)\, e^{8ik^3 t}. \tag{4.17}$$

In order to evaluate the time evolution of the dependency constants $\gamma_j(t)$, we can substitute $\gamma_j(t) f_r(i\kappa_j, x;t)$ for $f_l(i\kappa_j, x;t)$ [cf. (2.10)] and evaluate (4.13) at $k = i\kappa_j$. We get

$$f_r(i\kappa_j, x;t)\, \partial_t \gamma_j(t) + \gamma_j(t)\, \partial_t f_r(i\kappa_j, x;t) - \gamma_j(t) \mathcal{A} f_r(i\kappa_j, x;t) = -4\kappa_j^3 \gamma_j(t) f_r(i\kappa_j, x;t). \tag{4.18}$$

On the other hand, from (4.16) at $k = i\kappa_j$, we obtain

$$\gamma_j(t)\,\partial_t f_{\rm r}(i\kappa_j, x; t) - \gamma_j(t)\,\mathcal{A} f_{\rm r}(i\kappa_j, x; t) = 4\kappa_j^3 \gamma_j(t)\, f_{\rm r}(i\kappa_j, x; t). \tag{4.19}$$

Subtracting (4.19) from (4.18) we conclude that $\partial_t \gamma_j(t) = -8\kappa_j^3 \gamma_j(t)$, which leads to

$$\gamma_j(t) = \gamma_j(0)\, e^{-8\kappa_j^3 t} = \gamma_j\, e^{-8\kappa_j^3 t}. \tag{4.20}$$

Then, from (2.9) and (4.15) we obtain the time evolution of the norming constants $c_{{\rm l}j}(t)$ and $c_{{\rm r}j}(t)$ as

$$c_{{\rm l}j}(t) = c_{{\rm l}j}(0)\, e^{4\kappa_j^3 t} = c_{{\rm l}j} e^{4\kappa_j^3 t}, \quad c_{{\rm r}j}(t) = c_{{\rm r}j}(0)\, e^{-4\kappa_j^3 t} = c_{{\rm r}j} e^{-4\kappa_j^3 t}.$$

5. AKNS METHOD TO DERIVE THE KdV

In the previous section we have seen that the KdV arises as a compatibility condition in the Lax method. There are other methods to derive nonlinear PDEs that can be solved by the inverse scattering transform, i.e. by solving the inverse problem with the time-evolved scattering data for a corresponding linear differential equation. One of these methods was developed by Ablowitz, Kaup, Newel, and Segur, and it was first applied to the Sine-Gordon equation [23]. Here we outline the basic idea behind the method of Ablowitz, Kaup, Newel, and Segur (AKNS method, for short) and use it to derive the KdV equation.

Given a linear operator $\mathcal{X}$ associated with the system $v_x = \mathcal{X} v$, we are interested in finding another operator $\mathcal{T}$ (the operators $\mathcal{X}$ and $\mathcal{T}$ are said to form an *AKNS pair*) such that:

(i) The spectral parameter λ does not change in time.

(ii) The quantity $v_t - \mathcal{T} v$ must remain a solution to $v_x = \mathcal{X} v$.

(iii) The quantity $\mathcal{X}_t - \mathcal{T}_x + \mathcal{X}\mathcal{T} - \mathcal{T}\mathcal{X}$ must be a (matrix) multiplication operator.

Note that in general $\mathcal{X}$ contains the spectral parameter λ, and hence $\mathcal{T}$ also depends on λ as well. Usually, $\mathcal{X}$ and $\mathcal{T}$ are matrix-valued with entries depending on x, t, and λ. As the operator $\mathcal{A}$ in the Lax method determines the time evolution of solutions to $\mathcal{L}\psi = \lambda\psi$, in the AKNS method the operator $\mathcal{T}$ determines the time evolution of solutions to $v_x = \mathcal{X}v$ according to condition (ii) listed above.

As the following argument shows, for compatibility, we are forced to have

$$\mathcal{X}_t - \mathcal{T}_x + \mathcal{X}\mathcal{T} - \mathcal{T}\mathcal{X} = 0, \tag{5.1}$$

which leads to an integrable PDE and is in general nonlinear. From condition (ii) above we see that

$$(v_t - \mathcal{T}v)_x = \mathcal{X}(v_t - \mathcal{T}v),$$

or equivalently,

$$\begin{aligned} v_{tx} - \mathcal{T}_x v - \mathcal{T}v_x &= \mathcal{X}v_t - \mathcal{X}\mathcal{T}v \\ &= (\mathcal{X}v)_t - \mathcal{X}_t v - \mathcal{X}\mathcal{T}v \\ &= (v_x)_t - \mathcal{X}_t v - \mathcal{X}\mathcal{T}v \\ &= v_{xt} - \mathcal{X}_t v - \mathcal{X}\mathcal{T}v. \end{aligned} \tag{5.2}$$

We expect v to be smooth enough so that $v_{tx} = v_{xt}$. Let us replace $\mathcal{T}v_x$ by $\mathcal{T}\mathcal{X}v$ on the left-hand side in (5.2), from which we get $(\mathcal{X}_t - \mathcal{T}_x + \mathcal{X}\mathcal{T} - \mathcal{T}\mathcal{X})v = 0$, which in turn as a result of condition (iii) listed above gives us the compatibility condition (5.1).

Let us write the Schrödinger equation (1.2), by replacing the spectral parameter k^2 by λ, in the form of the first-order linear system $v_x = \mathcal{X}v$ by choosing

$$v = \begin{bmatrix} \psi_x \\ \psi \end{bmatrix}, \quad \mathcal{X} = \begin{bmatrix} 0 & u(x,t) - \lambda \\ 1 & 0 \end{bmatrix}.$$

We will construct $\mathcal{T}$ so that $\mathcal{T}$ and $\mathcal{X}$ will form an AKNS pair. Let us try

$$\mathcal{T} = \begin{bmatrix} \alpha & \beta \\ \xi & \eta \end{bmatrix},$$

where the entries α, β, ξ, and η may depend on x, t, and λ. The compatibility condition (5.1) leads to

$$\begin{bmatrix} 0 & u_t \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \alpha_x & \beta_x \\ \xi & \eta_x \end{bmatrix} + \begin{bmatrix} 0 & u-\lambda \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \xi & \eta \end{bmatrix} - \begin{bmatrix} \alpha & \beta \\ \xi & \eta \end{bmatrix} \begin{bmatrix} 0 & u-\lambda \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

or equivalently

$$\begin{bmatrix} -\alpha_x - \beta + \xi(u-\lambda) & u_t - \beta_x + \eta(u-\lambda) - \alpha(u-\lambda) \\ -\xi_x + \alpha - \eta & -\eta_x + \beta - \xi(u-\lambda) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.3}$$

From the $(1,1)$, $(2,1)$ and $(2,2)$ entries of the above matrix equation, we obtain

$$\beta = -\alpha_x + (u-\lambda)\xi, \quad \eta = \alpha - \xi_x, \quad \eta_x = -\alpha_x. \tag{5.4}$$

Then the $(1,2)$ entry in (5.3) is given by

$$u_t + \frac{1}{2}\xi_{xxx} - u_x\xi - 2\xi_x(u-\lambda) = 0. \tag{5.5}$$

Letting $\xi = \lambda\zeta + \mu$ in (5.5), where λ is the spectral parameter, we obtain

$$2\zeta_x\lambda^2 + \left(\frac{1}{2}\zeta_{xxx} - 2\zeta_x u + 2\mu_x - u_x\zeta\right)\lambda + \left(u_t + \frac{1}{2}\mu_{xxx} - 2\mu_x u - u_x\mu\right) = 0.$$

Equating the coefficients of each power of λ to zero, we get

$$\zeta = c_1, \quad \mu = \frac{1}{2}c_1 u + c_2, \quad u_t - \frac{3}{2}c_1 u u_x - c_2 u_x + \frac{1}{4}c_1 u_{xxx} = 0, \tag{5.6}$$

where c_1 and c_2 are arbitrary constants. Using $c_1 = 4$ and $c_2 = 0$, from (5.6) we obtain the KdV given in (1.1). Moreover, with the help of (5.4) we get

$$\alpha = u_x + c_3, \quad \beta = -4\lambda^2 + 2\lambda u + 2u^2 - u_{xx}, \quad \xi = 4\lambda + 2u, \quad \eta = c_3 - u_x.$$

Letting $c_3 = 0$, we obtain

$$\mathcal{T} = \begin{bmatrix} u_x & -4\lambda^2 + 2\lambda u + 2u^2 - u_{xx} \\ 4\lambda + 2u & -u_x \end{bmatrix}. \tag{5.7}$$

It is possible to obtain the time evolution of the Jost solutions and of the scattering data via the AKNS method. Note that condition (ii) of the AKNS method is equivalent to having

$$v_t - \mathcal{T} v = p(t, \lambda)\, v, \tag{5.8}$$

for some (scalar) coefficient $p(t, \lambda)$. For example, if we choose

$$v = \begin{bmatrix} f_l'(k, x; t) \\ f_l(k, x; t) \end{bmatrix},$$

by letting $x \to +\infty$ in (5.8) and by using (2.2), we obtain the time evolution of $f_l(k, x; t)$ given in (4.13). By using (2.4) with $T(x)$ replaced by $T(k; t)$ and with $L(k)$ by $L(k; t)$, and by letting $x \to -\infty$ in (5.8) and using (2.2), we get the time evolutions given in (4.15). In a similar way, with the help of (2.3), (4.9), and (5.7), by choosing

$$v = \begin{bmatrix} f_r'(k, x; t) \\ f_r(k, x; t) \end{bmatrix},$$

and by letting $x \to \pm\infty$ in (5.8), we obtain (4.16) and (4.17).

6. SOLUTION TO THE CAUCHY PROBLEM FOR THE KdV

The Cauchy problem (initial-value problem) for the KdV consists of finding $u(x, t)$ when $u(x, 0)$ is known. As shown by Gardner, Greene, Kruskal, and Miura [11], this problem can be solved by using the inverse scattering transform. Let $\mathcal{D}(t)$ denote any of the equivalents of the scattering data for the the Schrödinger equation with the time-evolved potential $u(x, t)$. In other words, we have

$$\begin{aligned} \mathcal{D}(t) :&= \{R(k; t), L(k; t), T(k; t), \{\kappa_j(t)\}, \{\gamma_j(t)\}, \{c_{lj}(t)\}, \{c_{rj}(t)\}\} \\ &= \{R(k)\, e^{8ik^3 t}, L(k)\, e^{-8ik^3 t}, T(k), \{\kappa_j\}, \{\gamma_j\, e^{-8\kappa_j^3 t}\}, \{c_{lj}\, e^{4\kappa_j^3 t}\}, \{c_{rj}\, e^{-4\kappa_j^3 t}\}\}. \end{aligned} \tag{6.1}$$

Note that $\mathcal{D}(0)$ corresponds to the initial scattering data associated with the potential $V(x)$, where $u(x, 0) = V(x)$.

Below we outline the solution to the Cauchy problem for the KdV.

(i) Given $u(x,0)$, determine the corresponding scattering data $\mathcal{D}(0)$. This is done by solving the *direct scattering problem* $V(x) \mapsto \mathcal{D}(0)$. The solution to this problem is essentially equivalent to solving (2.1) and obtaining the Jost solutions $f_l(k,x)$ and $f_r(k,x)$, from which $\mathcal{D}(0)$ can be constructed.

(ii) Evolve in time the scattering data as $\mathcal{D}(0) \mapsto \mathcal{D}(t)$ in accordance with (6.1). Note that the time-evolution of the scattering data is really simple. On the other hand, the time evolution of the potential $u(x,0) \mapsto u(x,t)$ will be much more complicated. Similarly, we expect that the time evolution of the Jost solutions $f_l(k,x) \mapsto f_l(k,x;t)$ and $f_r(k,x) \mapsto f_r(k,x;t)$, governed by the PDEs (4.13) and (4.16), respectively, will be complicated.

(iii) Having obtained the time-evolved scattering data $\mathcal{D}(t)$, solve the corresponding inverse scattering problem $\mathcal{D}(t) \mapsto u(x,t)$ for (1.2). This problem is known to be uniquely solvable [13] when the initial potential V belongs to the Faddeev class.

As indicated in Section 3, in step (iii) above one can obtain the solution to the inverse scattering problem by solving, for example, the time-evolved Riemann-Hilbert problem [cf. (3.1)]

$$m_l(-k,x;t) = T(k)\, m_r(k,x;t) - R(k)\, e^{2ikx+8ik^3t}\, m_l(k,x;t), \qquad k \in \mathbf{R}, \tag{6.2}$$

and recover $u(x,t)$ by using [cf. (3.4)]

$$u(x,t) = \frac{m_l''(k,x;t)}{m_l(k,x;t)} + 2ik\frac{m_l'(k,x;t)}{m_l(k,x;t)}, \qquad k \in \overline{\mathbf{C}^+}, \tag{6.3}$$

where the right-hand side can be evaluated at any k value, including $k=0$ and $k=\pm\infty$. Equivalently, one can solve the time-evolved Riemann-Hilbert problem [cf. (3.2)]

$$m_r(-k,x;t) = T(k)\, m_l(k,x;t) - L(k)\, e^{-2ikx-8ik^3t}\, m_r(k,x;t), \qquad k \in \mathbf{R},$$

and use [cf. (3.4)]

$$u(x,t) = \frac{m_r''(k,x;t)}{m_r(k,x;t)} - 2ik\frac{m_r'(k,x;t)}{m_r(k,x;t)}, \qquad k \in \overline{\mathbf{C}^+},$$

where the right-hand side can be evaluated at any k value in $\overline{\mathbf{C}^+}$.

Alternatively, one can solve the time-evolved left Marchenko equation [cf. (3.5)]

$$B_\mathrm{l}(x,y;t)+\Omega_\mathrm{l}(2x+y;t)+\int_0^\infty dz\,\Omega_\mathrm{l}(2x+y+z;t)\,B_\mathrm{l}(x,z;t)=0,\qquad y>0,\qquad (6.4)$$

with

$$\Omega_\mathrm{l}(y;t):=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,R(k)\,e^{8ik^3t+iky}+\sum_{j=1}^{N}c_{\mathrm{l}j}^2\,e^{8\kappa_j^3t-\kappa_jy},$$

and recover $u(x,t)$ by using [cf. (3.6)]

$$u(x,t)=-2\,\frac{\partial B_\mathrm{l}(x,0^+;t)}{\partial x}.\qquad (6.5)$$

Equivalently, one can solve the time-evolved right Marchenko equation [cf. (3.8)]

$$B_\mathrm{r}(x,y;t)+\Omega_\mathrm{r}(-2x+y;t)+\int_0^\infty dz\,\Omega_\mathrm{r}(-2x+y+z;t)\,B_\mathrm{r}(x,z;t)=0,\qquad y>0,\quad (6.6)$$

with

$$\Omega_\mathrm{r}(\alpha;t):=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,L(k)\,e^{-8ik^3t+iky}+\sum_{j=1}^{N}c_{\mathrm{r}j}^2\,e^{-8\kappa_j^3t-\kappa_jy},$$

and obtain $u(x,t)$ by using [cf. (3.9)]

$$u(x,t)=2\,\frac{\partial B_\mathrm{r}(x,0^+;t)}{\partial x}.\qquad (6.7)$$

7. SOLITON SOLUTIONS TO THE KdV

Consider the Cauchy problem for the KdV corresponding to the initial scattering data with zero reflection coefficients, N bound states at $k=i\kappa_j$, and dependency constant γ_j, where we have the ordering $0<\kappa_1<\cdots<\kappa_N$. When $R\equiv 0$, from (2.11) we see that the transmission coefficient is given by

$$T(k)=\prod_{j=1}^{N}\frac{k+i\kappa_j}{k-i\kappa_j}.\qquad (7.1)$$

In this case, the inverse scattering problem $\mathcal{D}(t) \mapsto u(x,t)$ can be solved algebraically in a closed form, and the resulting solution $u(x,t)$ to the KdV is known as the N-soliton solution. Using (7.1) we can write (6.2) as

$$m_{\mathrm{l}}(-k,x;t)\prod_{j=1}^{N}(k-i\kappa_j)=m_{\mathrm{r}}(k,x;t)\prod_{j=1}^{N}(k+i\kappa_j),\qquad k\in\mathbf{R}. \tag{7.2}$$

From the analyticity properties of m_{l} and m_{r} it follows that each side in (7.2) is entire in k with a polynomial growth of leading term k^N as $k\to\infty$ in the complex plane $\mathbf{C}$. Further, since k appears as ik the Faddeev functions, both sides in (7.2) must be a polynomial of the form

$$k^N+ik^{N-1}a_{N-1}(x,t)+\cdots+i^N a_0(x,t), \tag{7.3}$$

where $a_j(x,t)$ are real valued and to be determined by using [cf. (2.10), (3.3), and (4.20)]

$$\gamma_j\, e^{2\kappa_j x-8\kappa_j^3 t}=\frac{m_{\mathrm{l}}(i\kappa_j,x;t)}{m_{\mathrm{r}}(i\kappa_j,x;t)},\qquad j=1,\dots,N. \tag{7.4}$$

From (7.2) and (7.3) we get

$$m_{\mathrm{l}}(k,x;t)=\frac{k^N-ik^{N-1}a_{N-1}(x,t)+\cdots+(-i)^N a_0(x,t)}{\prod_{j=1}^{N}(k+i\kappa_j)}, \tag{7.5}$$

$$m_{\mathrm{r}}(k,x;t)=\frac{k^N+ik^{N-1}a_{N-1}(x,t)+\cdots+i^N a_0(x,t)}{\prod_{j=1}^{N}(k+i\kappa_j)}. \tag{7.6}$$

Let us define

$$\omega_j:=\gamma_j\, e^{2\kappa_j x-8\kappa_j^3 t},\qquad j=1,\dots,N, \tag{7.7}$$

where we recall that the sign of ω_j is the same [cf. (2.10)] as that of $(-1)^{N-j}$. Using (7.5)-(7.7) in (7.4) we get a system of linear algebraic equations for the N unknowns $a_j(x,t)$, namely

$$\omega_j=\frac{\kappa_j^N-\kappa_j^{N-1}a_{N-1}(x,t)+\cdots+(-1)^N a_0(x,t)}{\kappa_j^N+\kappa_j^{N-1}a_{N-1}(x,t)+\cdots+a_0(x,t)},\qquad j=1,\dots,N,$$

which can be written as

$$\begin{bmatrix} \kappa_1^{N-1}(\omega_1+1) & \kappa_1^{N-2}(\omega_1-1) & \cdots & \omega_1-(-1)^N \\ \kappa_2^{N-1}(\omega_2+1) & \kappa_2^{N-2}(\omega_2-1) & \cdots & \omega_2-(-1)^N \\ \vdots & \vdots & \ddots & \vdots \\ \kappa_N^{N-1}(\omega_N+1) & \kappa_N^{N-2}(\omega_N-1) & \cdots & \omega_N-(-1)^N \end{bmatrix} \begin{bmatrix} a_{N-1}(x,t) \\ a_{N-2}(x,t) \\ \vdots \\ a_0(x,t) \end{bmatrix} = \begin{bmatrix} \kappa_1^N(1-\omega_1) \\ \kappa_2^N(1-\omega_2) \\ \vdots \\ \kappa_N^N(1-\omega_N) \end{bmatrix}. \tag{7.8}$$

Let $M^{(N)}$ denote the coefficient matrix in (7.8) whose (j,n) entry is given as

$$M_{jn}^{(N)} := \kappa_j^{N-n}\left[\omega_j-(-1)^n\right], \qquad j,n=1,\dots,N. \tag{7.9}$$

From (7.8), via Cramer's rule, it is possible to extract $a_{N-j}(x,t)$ explicitly as

$$a_{N-j}(x,t) = -\frac{\det P^{(N-j)}}{\det M^{(N)}}, \qquad j=1,\dots,N,$$

where the $(N+1)\times(N+1)$ matrix $P^{(N-j)}$ is given by

$$P^{(N-j)} := \begin{bmatrix} 0 & Y^{(N,j)} \\ Q^{(N)} & M^{(N)} \end{bmatrix},$$

with $Q^{(N)}$ being the $N\times 1$ matrix whose jth row contains the entry $\kappa_j^N(1-\omega_j)$ for $j=1,\dots,N$; and $M^{(N)}$ being the $N\times N$ matrix given in (7.9); and $Y^{(N,j)}$ being the $1\times N$ matrix whose jth column contains the entry 1 and all the remaining entries being zero. For example, we have

$$P^{(N-1)} := \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \kappa_1^N(1-\omega_1) & \kappa_1^{N-1}(\omega_1+1) & \kappa_1^{N-2}(\omega_1-1) & \cdots & \omega_1-(-1)^N \\ \kappa_2^N(1-\omega_2) & \kappa_2^{N-1}(\omega_2+1) & \kappa_2^{N-2}(\omega_2-1) & \cdots & \omega_2-(-1)^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \kappa_N^N(1-\omega_N) & \kappa_N^{N-1}(\omega_N+1) & \kappa_N^{N-2}(\omega_N-1) & \cdots & \omega_N-(-1)^N \end{bmatrix}.$$

We note that $u(x,t)$ can be constructed from $a_{N-1}(x,t)$ alone or from $a_0(x,t)$ alone. For example, using (6.3) and (7.5) in the limit $k\to\pm\infty$, we see that

$$u(x,t) = 2\,\frac{\partial a_{N-1}(x,t)}{\partial x}, \tag{7.10}$$

or using (6.3) and (7.5) at $k=0$ we see that

$$u(x,t)=\frac{1}{a_0(x,t)}\frac{\partial^2 a_0(x,t)}{\partial x^2}. \tag{7.11}$$

In fact, using (7.5) in (6.3) we see that one can recover $u(x,t)$ by using any one of the following $(N+1)$ equations:

$$u(x,t)=\frac{1}{a_{N-j}(x,t)}\left[\frac{\partial^2 a_{N-j}(x,t)}{\partial x^2}+2\frac{\partial a_{N-j-1}(x,t)}{\partial x}\right], \qquad j=0,1,\dots,N,$$

where we have defined $a_{-1}(x,t):=0$ and $a_N(x,t):=1$.

Alternatively, we can obtain the N-soliton solution $u(x,t)$ via (6.5) by solving in a closed form the left Marchenko equation (6.4), which has a degenerate kernel thanks to the fact that $R\equiv 0$. Similarly, $u(x,t)$ can be obtained via (6.7) by solving in a closed form the right Marchenko equation (6.6), which has a degenerate kernel.

It is also possible to obtain the solutions to the Marchenko equations (6.4) and (6.6) in an algebraic manner, without really solving the integral equations themselves. Below we illustrate this for the recovery of $B_{\rm l}(x,y;t)$ and obtain $u(x,t)$ via (6.5). Using $R\equiv 0$ we can write (6.2) as

$$m_{\rm l}(-k,x;t)-1=[T(k)-1]\,m_{\rm r}(k,x;t)+[m_{\rm r}(k,x;t)-1], \qquad k\in\mathbf{R}.$$

Taking the Fourier transform of both sides with $\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,e^{iky}$ and exploiting the analyticity properties of $m_{\rm l}$ and $m_{\rm r}$, we obtain [cf. (3.7), (3.10)]

$$B_{\rm l}(x,y;t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,[T(k)-1]\,m_{\rm r}(k,x;t)\,e^{iky}, \qquad y>0. \tag{7.12}$$

The right-hand side of (7.12) can be evaluated as a residue integral along the semicircular arc that is the boundary of $\mathbf{C}^+$. Using (7.1) in (7.12), we get

$$B_{\rm l}(x,y;t)=i\sum_{j=1}^{N}[\text{Res}\,(T,i\kappa_j)]\,m_{\rm r}(i\kappa_j,x;t)\,e^{-\kappa_j y}, \qquad y>0. \tag{7.13}$$

Using (2.9) and (7.4) in (7.13), we obtain

$$B_l(x,y;t) = -\sum_{j=1}^{N} \varepsilon_j\, m_l(i\kappa_j, x; t)\, e^{-\kappa_j y}, \tag{7.14}$$

where we have defined

$$\varepsilon_j := c_{lj}^2 e^{-2\kappa_j x + 8\kappa_j^3 t}.$$

Using the time-evolved version of (3.7), from (7.14) we get

$$m_l(k,x;t) = 1 - \sum_{j=1}^{N} \frac{i\varepsilon_j}{k + i\kappa_j} m_l(i\kappa_j, x; t)\, e^{-\kappa_j y}. \tag{7.15}$$

Putting $k = i\kappa_n$ in (7.15) for $n = 1, \dots, N$, it is possible to recover the $m_l(i\kappa_j, x; t)$ by solving the linear algebraic system

$$\begin{bmatrix} 1+\frac{\varepsilon_1}{\kappa_1+\kappa_1} & \frac{\varepsilon_2}{\kappa_1+\kappa_2} & \cdots & \frac{\varepsilon_N}{\kappa_1+\kappa_N} \\ \frac{\varepsilon_1}{\kappa_2+\kappa_1} & 1+\frac{\varepsilon_2}{\kappa_2+\kappa_2} & \cdots & \frac{\varepsilon_N}{\kappa_2+\kappa_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\varepsilon_1}{\kappa_N+\kappa_1} & \frac{\varepsilon_2}{\kappa_N+\kappa_2} & \cdots & 1+\frac{\varepsilon_N}{\kappa_N+\kappa_N} \end{bmatrix} \begin{bmatrix} m_l(i\kappa_1, x; t) \\ m_l(i\kappa_2, x; t) \\ \vdots \\ m_l(i\kappa_N, x; t) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{7.16}$$

Let Γ denote the coefficient matrix in (7.16), namely, let its (n, j) entry be given by

$$\Gamma_{nj} = \delta_{nj} + \frac{\varepsilon_j}{\kappa_n + \kappa_j}, \qquad n, j = 1, \dots, N,$$

where δ_{nj} denote the Kronecker delta. Then, using (7.14) and (7.16), we obtain $B_l(x, 0^+; t)$ as the ratio of two determinants as

$$B_l(x, 0^+; t) = \frac{\det Z}{\det \Gamma},$$

where Z is the matrix defined as

$$Z := \begin{bmatrix} 0 & \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_N \\ 1 & 1+\frac{\varepsilon_1}{\kappa_1+\kappa_1} & \frac{\varepsilon_2}{\kappa_1+\kappa_2} & \cdots & \frac{\varepsilon_N}{\kappa_1+\kappa_N} \\ 1 & \frac{\varepsilon_1}{\kappa_2+\kappa_1} & 1+\frac{\varepsilon_2}{\kappa_2+\kappa_2} & \cdots & \frac{\varepsilon_N}{\kappa_2+\kappa_N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{\varepsilon_1}{\kappa_N+\kappa_1} & \frac{\varepsilon_2}{\kappa_N+\kappa_2} & \cdots & 1+\frac{\varepsilon_N}{\kappa_N+\kappa_N} \end{bmatrix}.$$

It can be shown that $\det Z = \partial_x(\det \Gamma)$ and hence with the help of (6.5), we get

$$u(x,t) = -2\frac{\partial}{\partial x}\left[\frac{1}{\det \Gamma}\frac{\partial(\det \Gamma)}{\partial x}\right]. \tag{7.17}$$

Some Mathematica notebooks based on (7.10), (7.11), and (7.17) for the evaluation of N-soliton solutions to the KdV and their animations are available at the author's web page [24].

We can use (7.10), (7.11), or (7.17) to analyze properties of solitons of the KdV. For example, when $N = 1$, from (7.10) or (7.11) we get the single soliton solution to (1.1) as

$$u(x,t) = -2\kappa_1^2 \operatorname{sech}^2\left(\kappa_1 x - 4\kappa_1^3 t + \sqrt{\ln \gamma_1}\right). \tag{7.18}$$

It is seen that the amplitude of this wave is $2\kappa_1^2$, it moves in the positive x direction with speed $4\kappa_1^2$, and the dependency constant γ_1 plays a role in the initial location of the soliton. The width of the soliton is inversely proportional to κ_1, which can be seen, e.g., by using the fact that $\int_{-\infty}^{\infty} \partial x \, \sqrt{-u(x,t)}$ is equal to $\sqrt{2}\pi$.

By exploiting the properties of one-soliton solutions to the KdV, one can show that as $t \to +\infty$, the N-soliton solution to the KdV resembles a train of N separate solitons each traveling with speed $4\kappa_j^2$. In this case, the KdV can be considered for all $t \in \mathbf{R}$ and it can be shown that each soliton emerges from the nonlinear interaction by experiencing only a change in the phase. For details, the reader is referred to [3,4,6,25].

8. CONCLUSION

In this section we will comment on three aspects of the KdV; namely, the time evolution stated in condition (ii) of the Lax method outlined in Section 4, the conserved quantities, and the Bäcklund transformation.

Some references incorrectly state the time evolution associated with the Lax method. For example, in (1.2.10) of [3, page 6] it is stated that the evolution of the solutions to (1.2) is given by $\partial_t \psi = \mathcal{A}\psi$ with $\mathcal{A}$ as in (4.7), instead of the correct statement (ii) of Section 4. The incorrectness of $\partial_t \psi = \mathcal{A}\psi$ can be demonstrated explicitly by an elementary example. Consider the 1-soliton solution to the KdV given in (7.18). Let us choose $\gamma_1 = \kappa_1 = 1$. A solution to (1.2) is obtained as $f_l(i, x; t)$. Let us call that solution ψ. We have

$$\psi = \frac{1}{2}\, e^{-4t} \operatorname{sech}(x - 4t), \quad u(x,t) = -2 \operatorname{sech}^2(x - 4t).$$

It can directly be verified that ψ satisfies (1.2) with $k = i$, and $(\partial_t - \mathcal{A})\psi = -4\psi$, as indicated by the correct time evolvement (4.13) with $k = i$; hence $\partial_t \psi \neq \mathcal{A}\psi$.

In (1.7.6) of [6, page 25] and (4) of [5, page 81] it is incorrectly stated that the evolution of the solutions to (1.2) is given by $\partial_t \psi - \mathcal{A}\psi = c\psi$ with $\mathcal{A}$ as in (4.7) and c is an arbitrary constant or a function of t. The incorrectness of this can be demonstrated, for example, by choosing ψ as $f_l(k, x; t) + f_r(k, x; t)$, i.e. the sum of the Jost solutions to (1.2). From (4.13) and (4.16) it follows that, with this choice of ψ, the equation $\partial_t \psi - \mathcal{A}\psi = c\psi$ would hold if and only if $(4ik^3 + c) f_r(k, x; t) - (4ik^3 - c) f_l(k, x; t) = 0$ for $x \in \mathbf{R}$ and $t > 0$, which is impossible due to the linear independence of $f_l(k, x; t)$ and $f_r(k, x; t)$ on $x \in \mathbf{R}$. Note that (4.4) has two linearly independent eigenfunctions for each $\lambda > 0$ and hence $\partial_t \psi - \mathcal{A}\psi$ in general is not expected to be a constant multiple of ψ.

Let us note that a potential in the Faddeev class need not even be continuous. On the other hand, from (1.1) we see that classical solutions to the KdV are thrice differentiable with respect to x. Informally speaking, the discontinuities that may be present in the initial value $u(x, 0)$ disappear and $u(x, t)$ becomes smoother for $t > 0$. On the other hand, even though $u(x, t)$ changes as t increases, certain integrals involving $u(x, t)$ with respect to x remain unchanged in time. Such quantities are known as conserved quantities for the KdV. They can either be obtained directly from (1.1) or from the expansion of $T(k)$ in powers of $1/k$ as $k \to \pm\infty$ by using the fact that $T(k; t)$ does not change in time [cf. (4.15)]. When $u(x, t)$ is smooth, with the help of (2.6) we obtain

$$T(k; t) = 1 + \frac{C_1}{2ik} - \frac{C_1^2}{8k^2} + \frac{C_1^3}{48ik^3} - \frac{C_2}{8ik^3} + O(1/k^4), \qquad k \to \infty \text{ in } \overline{\mathbf{C}^+},$$

where we have defined

$$C_j := \int_{-\infty}^{\infty} dx\, u(x,t)^j, \qquad j \geq 1.$$

Thus, we have identified two of the infinite number of conserved quantities: i.e., C_1 and C_2 are independent of time and are equal to their values at $t = 0$. The time independence of C_1 and C_2 can also be obtained directly from (1.1). We can write (1.1) as $u_t = (3u^2 - u_{xx})_x$, and hence

$$\frac{dC_1}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} dx\, u(x,t) = 3u(x,t)^2 - u_{xx}(x,t)\Big|_{x=-\infty}^{\infty} = 0,$$

where we have used $u(x,t) = o(1)$ and $u_{xx}(x,t) = o(1)$ as $x \to \pm\infty$. Similarly, after multiplying (1.1) with $u(x,t)$, we can write the resulting equation as

$$(u^2)_t = (4u^3 - 2uu_{xx} + u_x^2)_x. \tag{8.1}$$

Integrating both sides of (8.1) on $x \in \mathbf{R}$, we get

$$\frac{dC_2}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} dx\, u(x,t)^2 = 4u(x,t)^3 - 2u(x,t)\, u_{xx}(x,t) + u_x(x,t)^2\Big|_{x=-\infty}^{\infty} = 0,$$

verifying the time independence of C_2.

Can we characterize the set of nonlinear PDEs solvable by an inverse scattering transform? In other words, can we find a set of necessary and sufficient conditions that guarantee an initial-value problem for a nonlinear PDE to be solvable via an inverse scattering transform related to a linear problem? There does not yet seem to be a satisfactory solution to the characterization problem. On the other hand, nonlinear PDEs solvable by an inverse scattering transform seem to have some common characteristic features such as the Lax pair, the AKNS pair, soliton solutions, an infinite number of conserved quantities, a Hamiltonian formalism, the Painlevé property, and the Bäcklund transformation. Here, we only briefly explain the last feature and refer the reader to [6] for details and other features.

A Bäcklund transformation is a means to produce another integrable nonlinear PDE from a given one. The basic idea is as follows. Assume v satisfies the integrable nonlinear

PDE $\mathcal{M}(v) = 0$, and u satisfies another integrable nonlinear PDE, say $\mathcal{Q}(u) = 0$. A relationship $\mathcal{P}(u,v) = 0$, which is called a Bäcklund transformation, involving v, u, and their derivatives allows us to obtain $\mathcal{Q}(u) = 0$ from $\mathcal{M}(v) = 0$. A Bäcklund transformation can also be used on the same nonlinear PDE to produce another solution from a given solution.

As an example, assume that v satisfies the *modified KdV* given by

$$v_t - 6v^2 v_x + v_{xxx} = 0, \qquad x \in \mathbf{R}, \quad t > 0. \tag{8.2}$$

Then, choosing

$$u = v_x + v^2, \tag{8.3}$$

one can show that

$$u_t - 6uu_x + u_{xxx} = (\partial_x + 2v)(v_t - 6v^2 v_x + v_{xxx}), \qquad x \in \mathbf{R}, \quad t > 0.$$

Thus, (8.2) and (8.3) imply (1.1). The Bäcklund transformation given in (8.3) is known as Miura's transformation [26]. For a Bäcklund transformation applied on the KdV to produce other solutions from a given solution, we refer the reader to [6,27].

Another interesting question is the determination of the linear problem associated with the inverse scattering transform. In other words, given a nonlinear PDE that is known to be solvable by an inverse scattering transform, can we determine the corresponding linear problem? There does not yet seem to be a completely satisfactory answer to this question. We mention that Wahlquist and Estabrook [28] developed the so-called *prolongation method* to derive the linear scattering problem associated with the KdV and refer the reader to [6] for details.

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