# MATH 337: Introduction to Soliton Theory

## Lecture II

### Inverse Scattering method for KdV equation

KdV equation has a very nice connection to the Schrodinger equation in Quantum mechanics. This equation is a special case of the "Sturm-Liouville" equation in mathematics. Using this equation we shall try to solve the initial value ~~of the~~ problem of the KdV equation. Indeed the Schrodinger equation is a part of the Lax equations. Before studying the Lax formalism we shall first see the how we use the Schrodinger equation corresponding to the KdV equation.

- Inverse Scattering and the KdV equation:

KdV equation:

$$u_t - 6uu_x + u_{xxx} = 0$$

Its connection with the Sturm-Liouville equation is established by letting

$$u = v_x + v^2$$

which is called the "Miura Transformation"

$$u_t = v_{tx} + 2vv_t = \left(\frac{\partial}{\partial x} + 2v\right) v_t$$

$$u_x = v_{xx} + 2vv_x$$

$$u_{xx} = v_{xxx} + 2v_x^2 + 2vv_{xx}$$

$$u_{xxx} = v_4 + 6v_x v_{xx} + 2v\,v_{xxx}$$

$$u_t - 6uu_x + u_{xxx} = \left(\frac{\partial}{\partial x} + 2v\right)v_t - 6(v_x + v^2)(v_{xx} + 2vv_x)$$
$$+ \left(\frac{\partial}{\partial x} + 2v\right) v_{xxx} + 6v_x v_{xx}$$
$$= \left(\frac{\partial}{\partial x} + 2v\right)(v_t + v_{xxx}) - 6v^2(v_{xx} + 2vv_x) - 12vv_x^2$$

$$\left(\frac{\partial}{\partial x} + 2v\right)(v^2 v_x) = 2vv_x^2 + v^2 v_{xx} + 2v^3 v_x$$

$$\Rightarrow \quad = \left(\frac{\partial}{\partial x} + 2v\right)(v_t + v_{xxx} - 6v^2 v_x) = 0$$

Modified KdV equation. Hence if $v$ solves the mKdV equation then $u$ solves the KdV equation

Miura transformation is quite important in studying the IST. of the KdV equation

let

$$v = \frac{\psi_x}{\psi}$$

then

$$u = \frac{\psi_{xx}}{\psi} - \frac{\psi_x^2}{\psi^2} + \frac{\psi_x^2}{\psi^2} = \frac{\psi_{xx}}{\psi}.$$

$$\Rightarrow \quad \psi_{xx} - u\,\psi = 0 \qquad (1)$$

Now since the KdV equation is invariant under the Gallilean transfn

$$u \to u - \lambda$$
$$x \to x + 6\lambda t \qquad \lambda \in \mathbb{R}$$
$$t \to t$$

$\Rightarrow$ (1) changes to

$$\psi_{xx} + (\lambda - u)\,\psi = 0 \qquad (2)$$

This is called the Sturm-Liuville equation.

If possible, find $\psi$ then this eqn. determines the function $u(x,t)$. This method is called the IST.

## The scattering problem

In order to solve the initial value problem

$$u_t - 6uu_x + u_{xxx} = 0$$

$$u(x,0) = u_0(x).$$

we shall use a method which is called "the inverse scattering method". This method first introduced to solve the initial value problem of the KdV equation which consists of four steps in general

1. Direct Problem: Finding the scattering variables
2. Time evolution of the scattering data.
3. Solving the Gelfand-Levitan-Marchenko eqn.
4. Determine $u(x,t)$

For the direct problem of the KdV eqn we shall require that $u(x)$ is integrable, i.e

$$\int_{-\infty}^{\infty} |u_0(x)|\,dx < \infty$$

$u(x)$ must decay sufficiently rapidly at infinity so that the Faddeev condition is also satisfied

$$\int_{-\infty}^{\infty} (1+|x|)\,|u_0(x)|\,dx < \infty$$

# Direct problem

## Scattering problem (Direct Problem)

Consider the differential equation

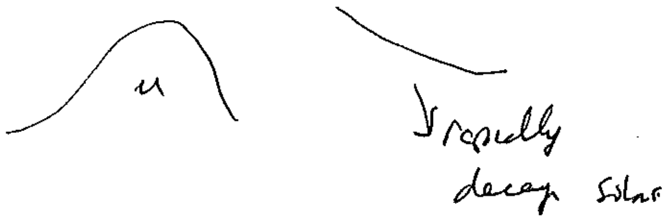
$$\psi_{xx} + (\lambda - u)\psi = 0$$

This equation is called Sturm-Liouville eigenvalue equation.

Let's forget the KdV equation for a while and consider the SL equation. This equation is the famous Schrodinger equation in Quantum mechanics. The idea there is to solve $\psi$ for given potential function $u$ ($t$ is a parameter here). Pictorially we have the following situation:

i) 


when $\lambda > 0$
continuous spectrum

ii) 


when $\lambda < 0$
discrete spectrum

Since asymptotically we require $u, u_x, \ldots \to 0$ as $|x| \to \infty$ then, asymptotically the SL equation becomes

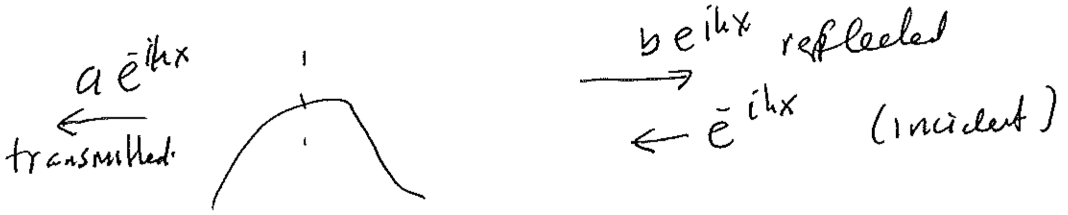
$$\psi_{xx} + \lambda \psi = 0 \qquad \text{as} \quad |x| \to \infty$$

Hence depending on the sign of $\lambda$ we have

a) continuous spectrum for $\lambda \geq 0$ $\quad \lambda = k^2$

$$\psi = e^{-ikx} + b(k) e^{ikx} \qquad x > 0$$

$$\psi = a\, e^{-ikx}$$




b = coefficient of reflection
a = transmission coefficient

b) Discrete Spectrum for $\lambda < 0$ $\quad \lambda = -k_n^2$

$n = 1, 2, \ldots, N$

$$\psi_n'' = k_n^2 \psi \quad \Rightarrow \quad \psi_n = \alpha_n e^{-k_n x} \qquad x > 0$$

$$\psi_n = \beta_n e^{+k_n x} \qquad x < 0$$

$\alpha_n, \beta_n$ are normalization condition

$$\int_{-\infty}^{\infty} \psi_n^2 \, dx = 1$$

Solutions must be continuous at $x=0$

$$\Rightarrow \alpha_1 = \beta_1$$

normalization condition

$$\alpha_n^2 \int_{-\infty}^{0} e^{2K_n x}\,dx + \alpha_n^2 \int_{0}^{\infty} e^{-2K_n x}\,dx = 1$$

$$\alpha_n^2 = \frac{1}{K_n} > 0 \qquad \alpha_n = \frac{1}{\sqrt{K_n}}$$

In general we have an eigenvalue problem

$$\psi'' + (\lambda - u)\psi = 0$$

for the discrete case we have

$$\psi_n'' = (K_n^2 + u)\psi_n$$

i) Lemma: eigenfunctions for different eigenvalues are orthogonal

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Conservation of energy

$$\psi'' + (k^2 - u)\psi = 0$$

$$\psi^{*\prime\prime} + (k^2 - u)\psi^* = 0$$

$$\Rightarrow \quad \frac{d}{dx} W = 0 \quad \Rightarrow \quad W(\psi, \psi^*) = \text{const.}$$

asymptotically

$$\psi = e^{-ikx} + b\, e^{ikx} \qquad x \to +\infty$$

$$\psi = a\, e^{-ikx} \qquad x \to -\infty$$

$$x \to \infty \qquad W = \psi \psi^{*\prime} - \psi^* \psi'$$

$$= \left(e^{-ikx} + b\, e^{ikx}\right)\left(+ik\, e^{ikx} - b^* e^{-ikx}\right)$$

$$- \left(-ik\, e^{-ikx} + ikb\, e^{ikx}\right)\left(\right.$$

$$= ik\,(1 - b^2) + ik\,(1 - b^2) = 2ik\,(1 - b^2)$$

$$x \to -\infty \qquad W = +2ik\, a^2$$

$$1 - b^2 = a^2$$

$$a^2 + b^2 = 1$$

$$\psi_n'' = (k_n^2 + u)\psi_n$$

$$\psi_m'' = (k_m^2 + u)\psi_m$$

$$\psi_m \psi_n'' - \psi_n \psi_m'' = (k_n^2 - k_m^2)\psi_n \psi_m$$

$$\frac{d}{dx}\left(\psi_m \psi_n' - \psi_n \psi_m'\right) = (k_n^2 - k_m^2)\psi_n \psi_m$$

$$\frac{d}{dx} W(\psi_m, \psi_n) = (k_n^2 - k_m^2)\psi_n \psi_m$$

integrate on $\mathbb{R}$

$$\int_{-\infty}^{\infty} \frac{d}{dx} W \, dx = W\Big|_{-\infty}^{\infty} = (k_n^2 - k_m^2)\int_{-\infty}^{\infty} \psi_n \psi_m \, dx$$

$W\Big|_{-\infty}^{\infty} = 0$ becan we are choosing decaying solutions

since $k_n \neq k_m \Rightarrow \int_{-\infty}^{\infty} \psi_n \psi_m \, dx = 0 \quad m \neq n$

If $k_n = k_m$ what about $\psi_n, \psi_m$ ?

$\frac{d}{dx} W = 0 \Rightarrow W = \text{const}$

but this constant $= 0$ $\quad W = 0$

$\Rightarrow \psi_m = \alpha \psi_n$ they normalized

hence $\alpha = \pm 1$.

Depending on $\lambda$ we have

i) continuum spectrum : scattering proble

ii) discrete spectrum. asymptotically decaying solutions , eigenvalues $\lambda = k^2$

$\lambda = -k_n^2$ , $n = 1, 2,$

Let's have some example

i). $u = -u_0 \delta(x)$ $u_0$ const.

$$\delta(x) = 0 \quad x \neq 0$$
$$= \infty \quad x = 0$$

$$\psi'' = -\lambda \psi \qquad x \neq 0$$

i) $\lambda = k^2$ (continuum spectrum). $-\infty < k < \infty$

$$\psi = e^{-ikx} + b\, e^{ikx} \qquad x > 0$$
$$\psi = a\, e^{-ikx} \qquad x < 0$$

$\psi$ is cont. at $x = 0$ $1 + b = a$

$\psi'$ has a jump at $x = 0$

$$\psi'' + \lambda \psi = -u_0 \delta(x) \psi$$
$$\int_{-\varepsilon}^{\varepsilon} \psi'' dx + \lambda \int_{-\varepsilon}^{\varepsilon} \psi = -u_0 \psi(0)$$

$$[\psi'] = -u_0 \psi(0)$$

(10)

$$\psi'_+ - \psi'_- = -u_0\,\psi(0)$$

$$-ik + ikb + ika = -u_0\, a = -$$

$$(a+b-1)\,ik = -u_0 a$$

$$\cancel{a(ik+u_0) = (1-b)\,ik}$$

$$\cancel{a = \frac{1-b}{ik+u_0}}$$

$$2(a-1)\,ik = -u_0 a$$

$$a(u_0 + 2ik) = 2ik$$

$$\boxed{a = \frac{2ik}{u_0+2ik} \qquad b = \frac{-u_0}{u_0+2ik}}$$

ii) discrete spectr. : $\lambda = -k_1^2$

$$\psi_{xx} = k_1^2\,\psi \qquad x \neq 0$$

$$\psi = \begin{cases} \alpha_1 e^{-k_1 x} & x>0 \\ \beta_1 e^{k_1 x} & x<0 \end{cases}$$

$\alpha_1 = \beta_1$ normalisch $\Rightarrow$ $\alpha_1^2 = \frac{1}{k_1}$

$$\psi_+' - \psi_-' = -u_0 \psi(0) = -u_0 \alpha_n$$

$$\alpha_n(-\kappa_n - \kappa_n) = -u_0 \alpha_n. \qquad \alpha_n \neq 0$$

$$\kappa_n = \frac{u_0}{2} \qquad \forall n$$

only one eigenvalue $n=1$.

Summary

i) $\lambda = k^2$ cont. spect

$$\psi = e^{-ikx} + b\, e^{ikx} \qquad x>0$$

$$= a\, e^{-ikx}$$

$$a = b+1, \qquad a = \frac{2ik}{u_0 + 2ik}$$

$$b = -\frac{u_0}{u_0 + 2ik}$$

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ii) $\lambda = -\kappa_n^2$

$$\psi = \frac{1}{\sqrt{\kappa_n}} e^{-\kappa_n x} \quad x>0$$

$$= \frac{1}{\sqrt{\kappa_n}} e^{\kappa_n x} \quad x<0$$

$$\kappa_n = \frac{u_0}{2} \qquad n=1.$$

one eigen value

. Scattering problem is completed.

Example: let $u = -u_0 \operatorname{sech}^2 x$ , $x \in \mathbb{R}$

$$\psi_{xx} + (\lambda + u_0 \operatorname{sech}^2 x)\psi = 0 \qquad (1)$$

let $y = \tanh x$ , then

$$\psi_x = \psi_y \operatorname{sech}^2 x$$

$$\psi_{xx} = \psi_{yy} \operatorname{sech}^4 x - 2\psi_y \frac{\sinh x}{\cosh^3 x}$$

$$= (1 - \tanh^2 x)^2 \psi_{yy} - 2\tanh x\,(1 - \tanh^2 x)\psi_y$$

$$= (1-y^2)^2 \psi_{yy} - 2y(1-y^2)\psi_y = (1-y^2)\frac{d}{dy}(1-y^2)\frac{d\psi}{dy}$$

Then (1) becomes

$$(1-y^2)\frac{d}{dy}(1-y^2)\frac{d\psi}{dy} + (\lambda + u_0(1-y^2))\psi = 0$$

or

$$\frac{d}{dy}(1-y^2)\frac{d\psi}{dy} + \left[u_0 + \frac{\lambda}{1-y^2}\right]\psi = 0$$

This equation is the well-known "Associated Legendre" equation

a) If $u_0 = N(N+1)$, $N$ is a positive integer, We have discrete spectrum with
$\lambda = -k_n^2$, $y \in [-1, 1]$

$$\frac{d}{dy}(1-y^2)\frac{d\psi_n}{dy} + \left[ N(N+1) - \frac{k_n^2}{1-y^2} \right] \psi_n = 0$$

$$k_n = n = 1, 2, \ldots, N$$

$$\psi_n(y) = \alpha_n P_N^{\,n}(y) \quad , \quad \text{"Associated Legendre Polynomials"}$$

where

$$P_N^{\,n}(y) = (-1)^n (1-y^2)^{n/2} \frac{d^n}{dy^n} \cancel{(y^2-1)} P_N(y)$$

$$P_N(y) = \frac{1}{2^N N!} \frac{d^N}{dy^N} (y^2-1)^N \quad , \quad \text{"Legendre Polynomials"}$$

$N = 1$ $\quad P_1(y) = y$

$$P_1^1(y) = -(1-y^2)^{1/2} = -\operatorname{sech} x$$

$N=2 \qquad P_2(y) = \frac{1}{8}\frac{d^2}{dy^2}(y^2-1)^2 = \frac{1}{2}(3y^2-1)$

$n=1 \qquad P_2^1(y) = -3(1-y^2)^{1/2}y. = -3\,\mathrm{sech}\,x\,\tanh x$

$n=2 \qquad P_2^2(y) = (1-y^2)\frac{d^2}{dy^2}P_2 = 3(1-y^2) = 3\,\mathrm{sech}^2 x$

$\psi_1 = -3\alpha_1\,\mathrm{sech}\,x\,\tanh x, \qquad \lambda_1 = 1$

Normalization $\Rightarrow \quad (3\alpha_1)^2 = 3/2$

$\psi_2 = 3\alpha_2\,\mathrm{sech}^2 x, \qquad \lambda_2 = 2$

normalization $\Rightarrow \quad (3\alpha_2)^2 = \frac{3^{1/2}}{2}$

b) Continuous spectrum: $\lambda > k^2 > 0$

$$\frac{d}{dy}\left((1-y^2)\frac{d\psi}{dy}\right) + \left(u_0 + \frac{k^2}{1-y^2}\right)\psi = 0$$

Let $\quad \psi = (1-y^2)^{-ik/2}\,\tilde{\psi}(y)$

$$\Psi_y = iky\,(1-y^2)^{-\frac{ik}{2}-1}\,\tilde{\varphi} + (1-y^2)^{-ik/2}\,\tilde{\varphi}_y$$

$$\Psi_{yy} = ik\,(1+y^2 + iky^2)\,(1-y^2)^{-\frac{ik}{2}-2}\,\tilde{\varphi}$$
$$+ 2iky(1-y^2)^{-\frac{ik}{2}-1}\,\tilde{\varphi}_y + (1-y^2)^{-ik/2}\,\tilde{\varphi}_{yy}$$

$$-2y\,\Psi_y + (1-y^2)\,\Psi_{yy} + \left(u_0 + \frac{k^2}{1-y^2}\right)\Psi = 0$$

$$\Rightarrow \quad -2ik\,y^2(1-y^2)^{-\frac{ik}{2}-1}\,\tilde{\varphi} - 2y\,(1-y^2)^{-ik/2}\,\tilde{\varphi}_y$$

$$+ik\,(1+y^2 + iky^2)(1-y^2)^{-\frac{ik}{2}-1}\,\tilde{\varphi} + 2iky(1-y^2)^{-\frac{ik}{2}}\,\tilde{\varphi}_y$$
$$+(1-y^2)^{-\frac{ik}{2}+1}\,\tilde{\varphi}_{yy} + \left(u_0 + \frac{k^2}{1-y^2}\right)(1-y^2)^{-ik/2}\,\tilde{\varphi} = 0$$

$\Rightarrow$

$$(1-y^2)\,\tilde{\varphi}_{yy} + 2y\,(-1+ik)\,\tilde{\varphi}_y + (k^2 - ik + u_0)\,\tilde{\varphi} = 0$$

let

$$z = \frac{1}{2}(1+y) \quad \Rightarrow \quad y = 2z-1$$

$$\Rightarrow \quad z(1-z)\,\varphi_{zz} + [1 - ik + 2(-1+ik)z]\,\varphi_z + (k^2 - ik + u_0)\,\tilde{\varphi} = 0$$

Hypergeometric equation

$$z(1-z)\tilde{\Psi}_{zz} + [c - z(a+b+1)]\tilde{\Psi}_z - ab\tilde{\Psi} = 0$$

Soln is the hypergeometric function

$$\Psi \sim F(a,b,c;z).$$

Here

$$a = \frac{1}{2} - ik + (u_0 + 1/4)^{1/2}$$

$$b = \frac{1}{2} - ik - (u_0 + 1/4)^{1/2}$$

$$c = 1 - ik$$

Solution:

$$\Psi(x,k) = a(k)\, 2^{-ik} (\operatorname{sech} x)^{-ik} F(a,b,c,\tfrac{1}{2}(1+y))$$

$$y = \tanh x$$

as $\quad X \longrightarrow \infty \qquad z = 1^+$

$\quad X \longrightarrow -\infty \qquad z = 1^-$

Then as $x \to \infty$

$$\psi \cong a(k)\frac{\Gamma(c)\,\Gamma(a+b-c)}{\Gamma(a)\Gamma(c)}\,e^{-ikx} + a(k)\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}\,e^{ikx}$$

Whereas $x \to -\infty$

$$\psi \cong \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(a+b-c)}\,e^{-ikx}$$

$$\Rightarrow \qquad a(k) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c)\,\Gamma(a+b-c)}$$

$$b(k) = a(k)\,\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} = \frac{\Gamma(a)\Gamma(b)\,\Gamma(c-a-b)}{\Gamma(a+b-c)\,\Gamma(c-a)\Gamma(c-b)}$$

<u>Remark 1:</u> $\Gamma(c-a)\,\Gamma(c-b)$

$$= \Gamma\left(1/2 - (u_0+1/4)^{1/2}\right)\Gamma\left(\tfrac{1}{2} + (u_0+1/4)^{1/2}\right)$$

$$= \frac{\pi}{\cos\pi(u_0+\frac{1}{4})^{1/2}}$$ "a property of the Gamma function"

$$\Rightarrow \quad b(k) = \frac{a(k)}{\pi}\,\Gamma(c)\,\Gamma(c-a-b)\,\cos\pi(u_0+1/4)^{1/2}$$

if $\quad (u_0 + 1/4)^{1/2} = (N + 1/2)$

$\Rightarrow b(h) = 0 \qquad$ reflectionless potentials

but

$$u_0 = -1/4 + (N + 1/2)^2 = N(N+1).$$

This is the potential for the discrete case. Hence reflectionless potentials has only discrete spectrum.

Remark 2. poles of $b(h)$: For instance if $b = -m$ where $m$ is a positive integer, then

$$-m = \frac{1}{2} - ik + (u_0 + 1/4)^{1/2}$$

$$ik = -m + 1/2 + (u_0 + 1/4)^{1/2}$$

so there exist finite number of eigenvalues if $(u_0 + 1/4)^{1/2} > \frac{1}{2}$ with $u_0 > 0$.

## II. Second step in the ISM for KdV equation.

Let us now go back to eigenvalue problem (or Sturm-Liouville equation)

$$\psi_{xx} + (\lambda - u(x,t))\psi = 0 \qquad (2)$$

here $t$ is just a parameter. Our purpose is to solve the initial value problem

$$u_t - 6uu_x + u_{xxx} = 0 \quad , \quad x \in \mathbb{R}$$

$$u(x,0) = u_0(x) \quad , \quad x \in \mathbb{R}$$

In the first step we found the scattering data $\{a(k), b(k), \alpha_n, k_n\}$ at $t=0$ by using (2) at $t=0$. Second step is to find the scattering data at any time $t$

$$\{a(k,t), b(k,t), \alpha_n(t), k_n(t)\} \qquad t > 0$$

For this purpose we need the time evolution of the eigenfunction $\psi$, $\psi_t = ?$

We let

$$\psi_t + u_x \psi - 2(u+2\lambda)\psi = R(x,t) \qquad (3)$$

and find $R$ by using (2) and (3). Hence we have

$$\psi_{xx} + (\lambda - u)\psi = 0 \qquad (4)$$

$$\psi_t + u_x \psi - 2(u+2\lambda)\psi_x = R(x,t) \qquad (5)$$

$$\psi_{tx} + u_{xx}\psi + u_x \psi_x - 2u_x \psi_x - 2(u+2\lambda)\psi_{xx} = R_x$$

$$\psi_{tx} + u_{xx}\psi - u_x \psi_x + 2(u+2\lambda)(\lambda-u)\psi = R_x$$

$$\psi_{tx} + [u_{xx} + 4\lambda^2 - 2u^2 - 2\lambda u]\psi - u_x \psi_x = R_x$$

$$\psi_{txx} + (u_{xxx} - 4uu_x - 2\lambda u_x)\psi$$
$$+ (\cancel{u_{xx}} + 4\lambda^2 - 2u^2 - 2\lambda u)\psi_x - \cancel{u_{xx}}\psi_x$$
$$+ u_x(\lambda - u)\psi = R_{xx}$$

$$\psi_{txx} + (4\lambda^2 - 2u^2 - 2\lambda u)\psi_x + (u_{xxx} - 5uu_x - \lambda u_x)\psi = R_{xx} \qquad (6)$$

but

$$\psi_{xxt} + (\lambda_t - u_t)\psi + (\lambda - u)\psi_t = 0$$

$$\psi_{xxt} = \overset{-\lambda_t\psi}{u_t\psi} - (\lambda-u)[-u_x\psi + 2(u+2\lambda)\psi_x + R] \qquad (7)$$

$\Rightarrow$ use (7) in (6) we get:

$$-\lambda_t\psi + u_t\psi + (\cancel{\lambda}-u)u_x\psi - 2(u+2\lambda)(\lambda-u)\psi_x - (\lambda-u)R$$
$$+ (4\lambda^2 - 2u^2 - 2\lambda u)\psi_x + (u_{xxx} - 5uu_x - \cancel{\lambda u_x})\psi = R_{xx}$$

$$(-\lambda_t + u_t - 6uu_x + u_{xxx})\psi + [-2u\lambda + 2u^2 - 4\lambda^2 + 4\lambda u + 4\lambda^2 - 2u^2$$
$$- 2\lambda u)\psi_x - (\lambda-u)R = R_{xx}$$

$$-\lambda_t\psi = (\lambda-u)R + R_{xx} = (\lambda-u)R + R_{xx}$$
$$= -\frac{\psi_{xx}}{\psi}R + R_{xx}$$

$$R_{xx}\psi - \psi_{xx}R = -\lambda_t\psi^2$$

$$(R_x\psi - \psi_x R)_x = -\lambda_t\psi^2$$

$$R_x\psi - \psi_x R\Big|_{-\infty}^{\infty} = -\lambda_t\int_{-\infty}^{\infty}\psi^2\,dx$$

$$0 = -\lambda_t N, \qquad N \neq 0 \;\exists\; \lambda_t = 0$$

$\lambda = $ constant $\Rightarrow$ we get $R = h(t)\psi$

$\left[ R_x \psi - \psi_x R = \text{constant } \forall x \in \mathbb{R} \text{ but constant} = 0 \right]$

$$\psi_t + u_x \psi - 2(u + 2\lambda)\psi_x = h(t)\psi$$

For any function $h(t)$ of $t > 0$.

Multiplying by $\psi$ we get

$$\frac{1}{2}(\psi^2)_t + u_x \psi^2 - 2(u+2\lambda)\psi\psi_x = h(t)\psi^2$$

$$\Rightarrow \quad \frac{1}{2}(\psi^2)_t + \left[ u\psi^2 - 2\psi_x^2 - 4\lambda\psi^2 \right]_x = h(t)\psi^2 \qquad (8)$$

$\Downarrow$

$u_x\psi^2 + 2u\psi\psi_x - 4\psi_x\psi_{xx} - 8\lambda\psi\psi_x$

$u_x\psi^2 + 2u\psi\psi_x + 4(\lambda - u)\psi\psi_x - 8\lambda\psi\psi_x$

$u_x\psi^2 - 2u\psi\psi_x - 4\lambda\psi\psi_x$ ✓

$\Rightarrow$ Integrating (8) over $(-\infty, \infty)$ we get

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} \psi^2 dx + \left[ u\psi^2 - 2\psi_x^2 - 4\lambda\psi^2 \right]\Big|_{-\infty}^{\infty} = h(t)\int_{-\infty}^{\infty}\psi^2 dx$$

For bound states (discrete spectrum) the second term in the LHS vanishes, hence

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}\psi^2\,dx = h(t)\int_{-\infty}^{\infty}\psi^2\,dx$$

$$\Downarrow$$

$$0 \qquad \int_{-\infty}^{\infty}\psi^2\,dx = \text{const}$$

Then $h(t)=0$, must vanish for discrete spectrum case ($\lambda<0$). For the continuum case it should be calculated.

We shall use the system of equations.

$$\psi_{xx} + (\lambda - u(x,t))\psi = 0$$

$$\psi_t + u_x\psi - 2(u+2\lambda)\psi_x = h(t)\psi$$

To find the time evolution of the scattering data. Here $h(t)=0$ for the discrete spectrum.

## B. Time evolution of scattering

i) Discrete ~~system~~ spectrum:

$u, u_x \to 0$ as $|x| \to \infty$.

$$\psi_n(x,t) \approx c_n(t) e^{-k_n x} \qquad \text{as } x \to \infty$$

$$\psi_t + u_x \psi - 2(u+2\lambda)\psi_x = 0, \quad \lambda = -k_n^2$$

$$\approx \psi_t + 4k_n^2 \psi_x \approx \dot{c}_n(t) e^{-k_n x} - 4k_n^3 e^{-k_n x} c_n \approx 0$$

$$\dot{c}_n(t) - 4k_n^3 c_n(t) \approx 0 \Rightarrow c_n(t) = c_n(0) e^{4k_n^3 t}$$

where $c_n(0)$ is normalization constant at $t=0$.

ii) Continuous spectrum: $\lambda = k^2 > 0$, and

$u, u_x \to 0$, as $|x| \to \infty$

$$\psi(x,t) = e^{-ikx} + b(k,t) e^{ikx} \qquad x \to \infty$$

$$\psi(x,t) = a(k,t) e^{-ikx} \qquad x \to -\infty$$

$$\psi_t - 4k^2 \psi_x = \beta(t) \psi \qquad \text{as } |x| \to \infty$$

as $x \to \infty$

$$\dot{b}\, e^{ikx} - 4k^2 \left(-ik e^{-ikx} + ik\, b e^{ikx}\right) = h\left(e^{-ikx} + b e^{ikx}\right)$$

$$\dot{b} - 4ik^3 b = h b.$$

$$4ik^3 = h.$$

$$\Rightarrow \quad \dot{b} = 8ik^3 b \Rightarrow b(k,t) = b(k,0)\, e^{8ik^3 t} \qquad t > 0$$

where $b(k,0)$ is the reflection coefficient at $t=0$.

as $x \to -\infty$

$$\dot{a}\, e^{-ikx} - 4k^2 \left(-ik\, a e^{-ikx}\right) = h a e^{-ikx}$$

since $h = 4ik^3 \Rightarrow \dot{a}(k,t) = 0$ or

$$a(k,t) = a(k,0) \qquad \forall\, t \geq 0$$

where $a(k,0)$ is the transmission coefficient at $t=0$.

Hence the scattering data $S(t)$ at any time is given by

$$S(t) = \{ k_n,\ c_n(t) = c_n(0)\, e^{4k_n^3 t},\ k,\ a(k,t) = a(k,0),\ b(k,t) = b(k,0)\, e^{8ik^3 t} \}$$

where

$$S(0) = \{ k_n,\ c_n(0),\ k,\ a_n(k,0),\ b(k,0) \}.$$

where we obtain $S(0)$ from the eigenvalue equation $\psi_{xx} + (\lambda - u_0(x))\psi = 0$ with $u_0(x)$ being the initial value of the KdV variable $u(x,0) = u_0(x)$.

The next step in the ISM is the inverse problem. Using the scattering data $S(t)$ find $u(x,t)$.

$$\begin{tikzcd}
u(x,0) \arrow[r, "\text{Direct problem}"] & S(0) \arrow[d, "\text{time evolution of the scattering data}"] \\
u(x,t) & S(t) \arrow[l, "\text{inverse problem}"', "\text{Gelfand-Levitan-Marchenko equation}"]
\end{tikzcd}$$

## Inverse Problem

If the scattering data $S(0)$ is known the solution of the KdV equation is given by the "Inverse scattering method". For this purpose the Last step is to solve the Gelfand-Levitan-Marchenko (GLM) equation.

$$K(x,z;t) + F(x+z;t) + \int_x^{\infty} K(x,y;t)F(y+z;t)\,dy = 0 \qquad (9)$$

where $K(x,y;t)$ is called the "kernel" of the above "Fredholm type" integral equation and $F(x,t)$ is given by

$$F(x,t) = \sum_{n=1}^{N} C_n^2(0)\, e^{8k_n^3 t - k_n x} + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k,0)\, e^{8ik^3 t + ikx}\, dk \qquad (10)$$

When we solve (9) for $K(x,z;t)$ then the solution of the KdV equation is

$$u(x,t) = -2\frac{\partial}{\partial x} K(x,x;t) \qquad (11).$$

The initial value problem of the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0 \quad , \quad x \in \mathbb{R}$$

$$u(x,0) = u_0(x) \quad , \quad x \in \mathbb{R}.$$

1) Let $u_0(x) = -2\,\text{sech}^2 x$ , where $U_0 = -2$ which corresponds to $N = 1$. Then

$\lambda_n = -k_n^2$ , $k_n = n$ , $n = 1$

we found that

$\psi_1(x) = \frac{1}{\sqrt{2}}\,\text{sech}\,x$ so that $\int_{-\infty}^{\infty} \psi_1^2\,dx = 1$

as $x \to \infty$ $\psi_1(x) = \frac{1}{\sqrt{2}} \frac{2}{e^x + e^{-x}} \sim \sqrt{2}\,e^{-x}.$

Therefore $C_1(0) = \sqrt{2} \Rightarrow C_1(t) = \sqrt{2}\,e^{4t}$

When $u_0 = -2$ reflection coefficient $b(k,0) = 0$

Hence

$$F(x,t) = \sum_{n=1}^{N} C_n^2(0)\,e^{8k_n^3 t - k_n x} = 2e^{8t - x} \qquad (12)$$

Inserting this into the GLM equation we obtain

$$K(x,z;t) + 2e^{8t-(x+z)} + 2e^{8t-z}\int_x^\infty K(x,y;t)e^{-y}\,dy = 0$$

Let

$$K(x,z;t) = L(x,t)\,e^{-z}$$

$\Rightarrow$

$$e^{-z}L(x,t) + 2e^{8t-(x+z)} + 2e^{8t-z}\int_x^\infty L(x,t)e^{-2y}\,dy = 0$$

$$\cancel{e^{-z}}L(x,t)\left[1 + e^{8t-2x}\right] = -2e^{8t-x-\cancel{z}}$$

$$L(x,t) = -\frac{2e^{8t-x}}{1+e^{8t-2x}}$$

Hence

$$K(x,z;t) = -\frac{2e^{8t-x-z}}{1+e^{8t-2x}}$$

$$K(x,x;t) = -2\frac{e^{8t-2x}}{1+e^{8t-2x}}$$

Since

$$u(x,t) = -2\frac{\partial}{\partial x}K(x,x;t)$$

We get

$$u(x,t) = 4\Big[-2e^{8t-2x}(1+e^{8t-2x}) + 2e^{8t-2x}\cdot e^{8t-2x}\Big](1+e^{8t-2x})^{-2}$$

$$= -8\,\frac{e^{8t-2x}}{(1+e^{8t-2x})^2}$$

$$= -8\,\frac{1}{[e^{-4t+x}+e^{4t-x}]^2}$$

$$u(x,t) = -2\,\mathrm{sech}^2(4t-x)$$

solitary wave solution (or one soliton solution) of the KdV equation

2) $u_0(x) = -6\,\mathrm{sech}^2 x$ , $U_0 = -6$ , $N=2$

$n=1 \Rightarrow K_1 = 1$ $\qquad \psi_1 = \sqrt{\frac{3}{2}}\,\tanh x\,\mathrm{sech}\,x$

$n=2 \Rightarrow K_2 = 2$ $\qquad \psi_2 = \frac{\sqrt{3}}{2}\,\mathrm{sech}^2 x$

as $x \to \infty$ $\qquad \tanh x \to 1$

$\qquad\qquad\qquad \operatorname{sech} x \to 2e^{-x}.$

$$\psi_1 \to 2\sqrt{3/2}\, e^{-x} \qquad C_1(0) = \sqrt{6}$$

$$\psi_2 \to \frac{\sqrt{3}}{2} \cdot 4 e^{-2x} \qquad C_2(0) = 2\sqrt{3}$$

Hence

$$F(x,t) = 6e^{8t-2x} + 12e^{64t-2x}.$$

Then GLM equation becomes

$$K(x,z;t) + 6e^{8t-(x+z)} + 12e^{64t-2(x+z)}$$

$$+ \int_x^\infty K(x,y;t)\left[6e^{8t-(y+z)} + 12e^{64t-(y+z)}\right]dy = 0$$

Let

$$K(x,z;t) = e^{-z} L_1(x,t) + e^{-2z} L_2(x,t)$$

$$\Rightarrow \quad L_1(x,t) + 6e^{8t-x} + 6e^{8t}\left[L_1(x,t)\int_x^\infty e^{-2y}dy\right.$$

$$\left. + L_2(x,t)\int_x^\infty e^{-3y}dy\right] = 0$$

$$L_2(x,t) + 12e^{64t-2x} + 12e^{64t}\left[L_1(x,t)\int_x^\infty e^{-3y}\,dy + \int_x^\infty e^{-4y}\,dy\right] = 0$$

or

$$L_1(x,t) + 6e^{8t-x} + 3L_1(x,t)e^{8t-2x} + 3L_2(x,t)e^{8t-3x} = 0$$

$$L_2(x,t) + 12e^{64t-2x} + 4L_1(x,t)e^{64t-3x} + 3L_2(x,t)e^{64t-4x} = 0$$

$$L_1(x,t) = \frac{6\left(e^{72t-5x} - e^{8t-x}\right)}{D}$$

$$L_2(x,t) = -12\,\frac{\left(e^{64t-2x} + e^{72t-4x}\right)}{D}$$

where

$$D = 1 + 3e^{8t-2x} + 3e^{64t-4x} + e^{72t-6x}$$

$\Rightarrow$

$$K(x,y;t) = \frac{6e^{-z}\left(e^{72t-5x} - e^{8t-x}\right)}{D} - 12\, e^{-2z}\frac{\left(e^{64t-2x} + e^{72t-4x}\right)}{D}.$$

$$K(x,x;t) = \frac{6\left(e^{72t-6x} - e^{8t-2x}\right)}{D} - 12\frac{\left(e^{64t-4x} + e^{72t-6x}\right)}{D}$$

$$= -\frac{6e^{72t-6x} + 12e^{64t-4x} + 6e^{8t-2x}}{1 + 3e^{8t-2x} + 3e^{64t-4x} + e^{72t-6x}}$$

$$u(x,t) = -2\frac{\partial}{\partial x} K(x,x;t)$$

$$= -12\frac{3 + 4\cosh(2x-8t) + \cosh(4x-64t)}{\left[3\cosh(x-28t) + \cosh(3x-36t)\right]^2}$$

This is the "two soliton" solution of the KdV equation