Introduction to Soliton Theory

Spring 2012

Chapter 1. The Korteweg-de Vries Equation

one dimensional wave equation (one dimensional string) two dimensional wave equation (membrane) four dimensional waves (electromagnetic waves)

As a simple example let us consider the one dimensional wave equation

$$c^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \qquad (1)$$

where $u(x,t)$ is the amplitude of the wave and c is a positive constant which represent the speed of the wave. General solution

$$u(x,t) = f(x-ct) + g(x+ct) \qquad (2)$$

where f and g are arbitrary functions. Here t denotes "time" and x denotes "space" coordinates. The functions f and g are determined if the initial conditions $u(x,0)$ and $u_t(x,0)$ are given. In this case we obtain the d'Alembert's solution

Let the initial condition be given by

$$u(x,0) = F(x)$$
$$u_t(x,0) = G(x) \quad , \quad x \in \mathbb{R}.$$

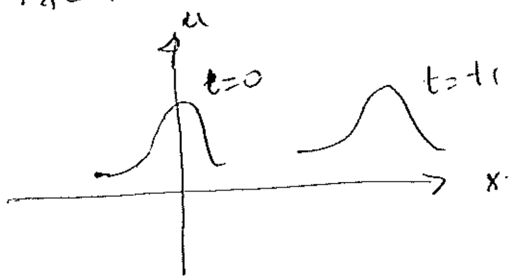
where F and G are given functions. Then the solution is (d'Alembert's solution).

$$u(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(x')\,dx' \qquad (3)$$

This solution describes two different waves moving opposite directions with the speed c. For instance the wave F(x-ct) moves toward right with velocity c and the wave G(x+ct) moves to the left with velocity c

Remark ⟹ These two waves do not interact. This is due to the "linearity of the wave" equation". Hence the solutions of the wave equation can be added (superposed). Since they do not interact they do not change their shape and speed as they propagate

To understant why the shape of the wave does not change, let us consider only one of the waves $F(x-ct)$. Let $\xi = x - ct$ (characteristic line). This wave to the right with speed c. When it moves to the right in Δt time it changes it's location by Δx amount keeping ξ fixed hence F remains fixed.



In order to study further properties of the waves we consider only one wave moving to one direction (let $G=0$).

- For more simpler case consider the wave equation

$$u_t + c u_x = 0 \Rightarrow u(x,t) = F(x-ct)$$

Such equations ($u_{tt} - c^2 u_{xx}$ and $u_t + cu_x = 0$) are obtained under the extremal assumptions under realistic conditions these equations turn out to be nonlinear. There are different types of waves dispersive, dissipative and nonlinear waves

1. <u>Dissipation</u>: Consider the wave

$$u_t + u_x + u_{xxx} = 0 \qquad (4)$$

To see how it behaves we let

$$u(x,t) = e^{i(kx - \omega t)} \qquad (5).$$

"harmonic wave solution" which is the starting point of the Fourier transform method.

5 Solves (4) if

$$\omega = k - k^3 \qquad (6)$$

This is called the dispersion relation. For all linear wave equation using the "harmonic wave" we obtain the dispersion relation $\omega = \omega(k)$.

Hence

$$kx - \omega t = k\,[x - (1-k^2)\,t] \qquad (7)$$

Here k is known as the "wave number", ω as the "frequency" of the wave. Then speed of the wave

$$c = \frac{\omega(k)}{k} = 1 - k^2 \qquad (8)$$

which is a function of k. Waves with different wave number propogate at different velocities. This the characteristic of the "dispersive waves".

The general solution is the Fourier transform of the harmonic solution

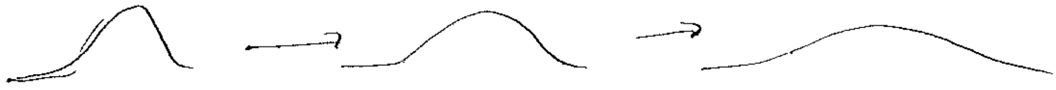
$$u(x,t) = \int_{-\infty}^{\infty} A(k) e^{i(kx - \omega(k)t)} \, dk. \qquad (9)$$

where the initial condition

$$u(x,0) = \int_{-\infty}^{\infty} A(k) e^{ikx} \, dk. \qquad (10)$$

Hence $A(k)$ is the Fourier transform of the initial condition $u(x,0)$.

A wave packet is composed of superposition of harmonic waves with different wave number, like in (9). Such wave packets change their shape as they move. In fact, since different components of the wave packet travel at different speeds the profile of the packet necessarily spread out or disperse

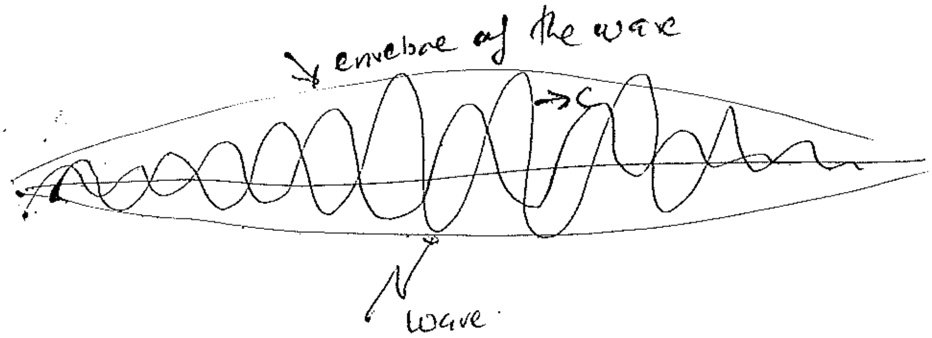


The velocity of an individual wave component is given by the equation $c = \frac{\omega}{k} = 1 - k^2$ and usually called the "phase velocity" of the wave. There is another velocity, defined as the group velocity,

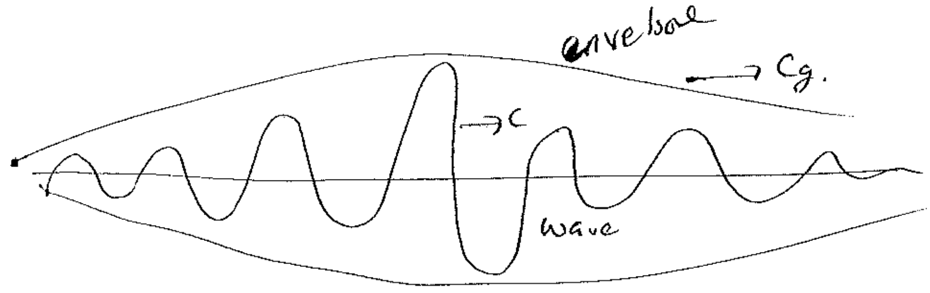
$$c_g = \frac{d\omega}{dk} = 1 - 3k^2 \qquad (11)$$

This is the velocity of ~~the~~ "wave packet" which is smaller than the phase velocity $c_g \le c$

Furthermore the group velocity is the velocity of propagation of energy (physical).



The wave moves with velocity c (phase velocity) the envelope moves with c_g.



2. Dissipation: In the previous example we considered the third order derivative in the wave equation. Let us see what happens if the wave equation contains an even number of x-derivatives: As an example consider the wave equation.

$$u_t + u_x + u_{xx} = 0 \quad (12)$$

This is the heat equation

let $u(x,t) = e^{i(kx-wt)}$, then the dispersion relation becomes complex

$$\omega = k - i k^2$$

Hence

$$u(x,t) = e^{-k^2 t} e^{ik(x-t)}$$

is a solution of (12). The wave moves with the speed unity for all k but it also decay exponentially for all k (real k)
(The sign of u_{xx} is important).

This decay is usually called "dissipation"

Remark: If we have combinations of the even and odd derivative terms in a wave eqn. the harmonic wave solutions both disperse and dissipate

3. <u>Nonlinearity</u>: In most cases we omit nonlinear terms in modelling wave phenomena. More realistic waves are nonlinear in character. For instance

$$u_t + (1+u)\,u_x = 0 \qquad (12)$$

Characteristic lines of this equation are the lines

$$x = (1+u)t + \text{const} \qquad (13).$$

<u>proof</u>:

$$\frac{dt}{1} = \frac{dx}{1+u} = \frac{du}{0}$$

$$Y_1 = u = \text{const.}$$

$$Y_2 = x - (1+u)t = \text{const.}$$

$$\Rightarrow \quad u(x,t) = f(Y_2) = f\,(x-(1+u)t)$$

$$u_t = f'\,[-u_t t - (1+u)]$$

$$u_x = f'\,[1 - u_x t]$$

$$u_t + (1+u)u_x = f'\,[-t u_t - \cancel{(1+u)} + \cancel{1+u} - t(1+u)u_x]$$

$$= -t f'\,[u_t + (1+u)u_x].$$

or $\quad [u_t + (1+u)u_x](1+tf') = 0$

since f is arbitrary then $u_t + (1+u)u_x = 0$

Take for instance

$$f = x^2$$

Then

$$u = (x-(1+u)t)^2$$

or

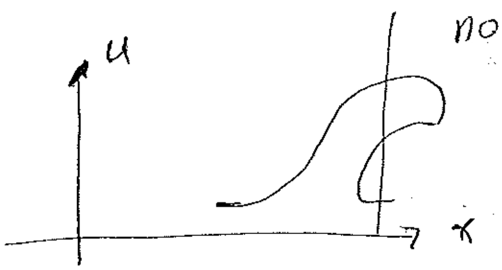
$$u^2 t^2 + u(2t^2 - 2xt - 1) + x^2 + t^2 - 2xt = 0$$

Non unique solutions.

$$u = \frac{-(2t^2-2xt-1) \pm \sqrt{(2t^2-2xt-1)^2 - 4t^2(x^2+t^2-2xt)}}{2t^2}$$

for $t > 0$

not single value (not a function)



wave fronts move faster due to the nonlinearity

- Superposition is lost. Hence we can not add two different solutions to obtain most general solution.

- The solutions obtain this way (of a nonlinear wave equation) change their shape

4. Wave equations having all properties:

By suitable assumptions in a given problem we can obtain equations which possesses both nonlinear and dispersive terms (or dissipative terms)

nonlinear + dispersive (KdV equation)

$$u_t + (1+u)u_x + u_{xxx} = 0 \qquad (14)$$

nonlinear + dissipative (Burgers equation).

$$u_t + (1+u)u_x - u_{xx} = 0 \qquad (15)$$

Burgers equation can be linearized by letting

$$1+u = -\frac{2\psi_x}{\psi}$$

Then

$$u_t = -\frac{2\psi_{tx}}{\psi} + \frac{2\psi_t\psi_x}{\psi^2}$$

$$u_x = -\frac{2\psi_{xx}}{\psi} + \frac{2\psi_x^2}{\psi^2}$$

$$u_{xx} = -\frac{2\psi_{xxx}}{\psi} + \frac{6\psi_{xx}\psi_x}{\psi^2} - \frac{4\psi_x^3}{\psi^3}$$

$$\Rightarrow \quad u_t + (1+u)u_x - u_{xx}$$

$$= -\frac{2\psi_{tx}}{\psi} + \frac{2\psi_t\psi_x}{\psi^2} - \frac{2\psi_x}{\psi}\left(-\frac{2\psi_{xx}}{\psi} + \frac{2\psi_x^2}{\psi^2}\right)$$

$$+\frac{2\psi_{xxx}}{\psi} - \frac{6\psi_{xx}\psi_x}{\psi^2} + \frac{4\psi_x^3}{\psi^3}$$

$$= -\frac{2}{\psi}(\psi_t - \psi_{xx})_x + \frac{2\psi_x}{\psi^2}(\psi_t - \psi_{xx})$$

$$= -2\left(\frac{\psi_t - \psi_{xx}}{\psi}\right)_x = 0 \Rightarrow \psi_t - \psi_{xx} = \lambda\psi \qquad (16)$$

where λ - is an arbitrary constant. $\lambda = 0$ case is the heat equation.

There are transformation which transform KdV equation to a linear equation such as the Burgers equation.

The KdV equation.

$$u_t + (1+u)u_x + u_{xxx} = 0. \qquad (17)$$

We can change it to a more proper one

$$\left.\begin{aligned} 1+u &= -6v \\ u_t &= -6v_t \\ u_x &= -6v_x \\ u_{xxx} &= -6v_{xxx} \end{aligned}\right\} \Rightarrow \quad v_t - 6vv_x + v_{xxx} = 0 \qquad (18)$$

Symmetries of the KdV equation: Finding transformation of the variables (v, x, t) so that the KdV equation is left unchanged

a) The scale transformation:

$$\left.\begin{aligned} x &= kX \\ t &= k^3 T \\ v &= k^{-2} U \end{aligned}\right\} \quad \text{Forms a group (of transformations) } G_k$$

$$G_k : (X, T, U) \longrightarrow (x, t, v)$$

i) $G_k G_\ell = G_{(k\ell)}$

$$(G_k G_\ell)(Y, \tau, V) = G_k(\ell Y, \ell^3 \tau, \ell^{-2} V)$$

$$= (k(\ell Y), k^3(\ell^3 \tau), k^{-2}(\ell^{-2} V))$$

$$= (x, t, v)$$

$x = (kl)X, \quad t = (kl)^3 T, \quad v = (kl)^{-2} V$

$\Rightarrow \quad G_k G_l = G_{(kl)}$

ii) G_1 is the unit element

iii) $G_{k^{-1}}$ is the inverse element of G_k.

iv) $G_k G_l = G_l G_k = G_{(kl)}$

commutative group

v) $G_k (G_l G_m) = (G_k G_l) G_m$

associative.

This group (a one parameter group) is also called a symmetry group of the KdV equation.

b) Translation : G_ε

$$\left.\begin{aligned} x &= X + \varepsilon \\ t &= T \\ v &= U \end{aligned}\right\} \quad \begin{aligned} &U_T - 6UU_X + U_{XXX} = 0 \\ &G_\varepsilon \end{aligned}$$

where ε is any real parameter

i) $G_{\varepsilon_1} G_{\varepsilon_2} = G_{\varepsilon_1 + \varepsilon_2}$

ii) G_0 is the unit element

iii) $G_{-\xi}$ is the inverse element of G_ξ

iv) Associativity is satisfied

$$G_{\xi_1}\,(G_{\xi_2}\cdot G_{\xi_3}) = (G_{\xi_1}\, G_{\xi_2})\, G_{\xi_3}$$

c) Time translation : G_η

$$\left.\begin{aligned} x &= X \\ t &= T + \eta \\ v &= U \end{aligned}\right\} \quad \begin{aligned} &U_T - 6UU_X + U_{XXX} = 0 \\ &G_\eta \end{aligned}$$

where η is any real parameter.
such transformations also form a group

i) $G_{\eta_1} G_{\eta_2} = G_{\eta_2} G_{\eta_1} = G_{\eta_1+\eta_2}$

ii) G_0 is the unit element

iii) $G_{-\eta}$ is the inverse of G_η

iv) Associativity is satisfied

$$G_{\eta_1}\,(G_{\eta_2}\, G_{\eta_3}) = (G_{\eta_1}\, G_{\eta_2})\, G_{\eta_3}$$

d) The Galileo transformation.

$t = T$ $\qquad T = t$

$x = X + 6\lambda T$ $\qquad X = x - 6\lambda t$

$V = U \mp \lambda$ $\qquad U = V + \lambda$

Where λ is any parameter.

$$\Rightarrow \quad V_t - 6VV_x + V_{xxx} = 0 \quad \Rightarrow$$

$$V_t = U_t = U_T + U_X 6\lambda$$

$$V_x = U_x = U_X$$

$$\Rightarrow \quad U_T \mp 6\lambda U_X - 6(U \mp \lambda)U_X + U_{XXX} = 0$$

$$U_T - 6UU_X + U_{XXX} = 0 \qquad \text{(KdV)}.$$

G_λ form also a group.

i) $G_{\lambda_1} G_{\lambda_2} = G_{\lambda_1 + \lambda_2}$

ii) G_0 is the unit element

iii) $G_{-\lambda}$ is the inverse element of G_λ.

iv) G_λ is associative

Group Invariant solutions:

The idea is to find invariant dependent and independent variables under each transformation. With respect to these variables KdV equation reduces to an ODE which may be solved easily.

i) Under the scale transformations G_k.

$$x = kX \,, \quad t = k^3 T \,, \quad v = k^{-2} U$$

invariant variables

$$\xi = \frac{x^3}{t} = \frac{X^3}{T}$$ new independent variable (or invariant independent variable)

$$\zeta(\xi) = x^2 v = X^2 U$$ (new dependent variable)

expressing $v(x,t)$ in terms of ζ and ξ we get

$$v(x,t) = \frac{1}{x^2}\,\zeta(\xi)$$

inserting this into the KdV equation

$$v_t - 16\, v v_x + v_{xxx} = 0$$

we get

$$v_t = \frac{1}{x^2}\,\zeta_{,\xi}\left(-\frac{x^3}{t^2}\right) = -\frac{x}{t^2}\,\zeta_{,\xi}$$

$$V_x = -\frac{2}{x^3}\zeta_1 + \frac{1}{x^2}\zeta_{1,\xi}\left(\frac{3x^2}{t}\right) = -\frac{2}{x^3}\zeta_1 + \frac{3}{t}\zeta_{1\xi}.$$

$$V_{xx} = \frac{6}{x^4}\zeta_1 - \frac{2}{x^3}\zeta_{1\xi}\left(\frac{3x^2}{t}\right) + \frac{3}{t}\zeta_{1\xi\xi}\left(\frac{3x^2}{t}\right)$$

$$= \frac{6}{x^4}\zeta_1 - \frac{6}{xt}\zeta_{1\xi} + \frac{9x^2}{t^2}\zeta_{1\xi\xi}.$$

$$V_{xxx} = -\frac{24}{x^5}\zeta_1 + \frac{6}{x^4}\zeta_{1\xi}\left(\frac{3x^2}{t}\right) + \frac{6}{x^2t}\zeta_{1\xi} - \frac{6}{xt}\zeta_{1\xi\xi}\left(\frac{3x^2}{t}\right)$$

$$+ \frac{18x}{t^2}\zeta_{1\xi\xi} + \frac{9x^2}{t^2}\zeta_{1\xi\xi\xi}\left(\frac{3x^2}{t}\right)$$

$$= -\frac{24}{x^5}\zeta_1 + \frac{24}{x^2t}\zeta_{1\xi} + \frac{27x^4}{t^3}\zeta_{1\xi\xi\xi}.$$

$\Rightarrow$ KdV equation $\Rightarrow$

$$-\frac{x}{t^2}\zeta_{1,\xi} - 6\frac{1}{x^2}\zeta_1\left(-\frac{2}{x^3}\zeta_1 + \frac{3}{t}\zeta_{1\xi}\right)$$

$$-\frac{24}{x^5}\zeta_1 + \frac{24}{x^2t}\zeta_{1\xi} + \frac{27x^4}{t^3}\zeta_{1\xi\xi\xi} = 0$$

multiplying by x^5 we get

$$-\xi^2\zeta_{1,\xi} + 12\zeta_1^2 - 24\zeta_1 - 18\xi^2\zeta_1\zeta_{1,\xi} + 24\xi\zeta_{1\xi} + 27\xi^3\zeta_{1\xi\xi\xi} = 0$$

This is a reduced equation.

We can obtain more simpler one if we take the invariant variables as

$$\xi = \frac{x}{t^{1/3}} \quad , \quad \zeta = t^{2/3} v$$

$$\Rightarrow v(x,t) = t^{-2/3} \zeta(\xi)$$

$$v_t = -\frac{2}{3} t^{-5/3} \zeta + t^{-2/3} \zeta_{,\xi} \left(-\frac{1}{3} \frac{x}{t^{4/3}}\right)$$

$$= -\frac{2}{3} t^{-5/3} \zeta - \frac{1}{3} \frac{x}{t^2} \zeta_{,\xi}$$

$$v_x = t^{-2/3} \zeta_{,\xi} \, t^{-1/3} = t^{-1} \zeta_{\xi} = t^{-1} \zeta_{,\xi}$$

$$v_{xxx} = t^{-5/3} \zeta_{\xi\xi\xi}$$

KdV equation $\Rightarrow$

$$-\frac{2}{3} t^{-5/3} \zeta - \frac{1}{3} \frac{x}{t^2} \zeta_{\xi} - 6 \frac{1}{t^{2/3}} \zeta \left(t^{-1} \zeta_{\xi}\right) + t^{-1} \zeta_{\xi\xi\xi} = 0$$

multiply by $t^{5/3}$ we get

$$-\frac{2}{3} \zeta - \frac{1}{3} \xi \zeta_{\xi} - 6 \zeta \zeta_{\xi} + \zeta_{\xi\xi\xi} = 0 \quad \checkmark$$

A reduced ODE

$$\zeta_1''' - 6\zeta_1\zeta_1' - \tfrac{1}{3}\xi\zeta_1' - \tfrac{2}{3}\zeta_1 = 0$$

let

$$\zeta_1 = -\tfrac{1}{6}\, y(\xi)$$

$$\Rightarrow \quad y''' + y y' - \tfrac{1}{3}\xi y' - \tfrac{2}{3} y = 0$$

let

$$y = \frac{dw}{d\xi} - \frac{1}{6} w^2$$

$$\Rightarrow \quad w'''' - \tfrac{1}{3} w w''' - \tfrac{1}{3} w w'^2 - \tfrac{1}{6} w^2 w_y''$$
$$+ \tfrac{1}{18} w^3 w' - \tfrac{1}{3}\xi w'' + \tfrac{1}{9}\xi w w'$$
$$- \tfrac{2}{3} w' + \tfrac{1}{9} w^2 = 0$$

This equation can be simplified

$$\left(D_\xi - \tfrac{1}{3} w\right)\left(w''' - \tfrac{1}{6} w^2 w' - \tfrac{1}{3}\xi w' - \tfrac{1}{3} w\right) = 0$$

$$\Rightarrow \quad w''' - \tfrac{1}{6} w^2 w' - \tfrac{1}{3}\xi w' - \tfrac{1}{3} w = 0$$

$$w''' - \tfrac{1}{18}(w^3)' - \tfrac{1}{3}(\xi w)' = 0$$

or $\quad w'' - \tfrac{1}{18} w^3 - \tfrac{1}{3}\xi w - k = 0, \quad k = \text{const}$

This is is the equation for second Painleve transcenden P_{II}.

ii) Time translation:

$$\left.\begin{aligned} t' &= t - t_0 \\ x' &= x \\ u' &= u \end{aligned}\right\}$$

invariant variables are x and u.
$\Rightarrow \; u = u(x)$

group invariant solutions

$$-6vv' + v''' = 0 \quad \Rightarrow \quad v'' - 3v^2 + k = 0$$

where k is an arbitrary constant.

iii) Space translation:

$$\left.\begin{aligned} t' &= t \\ x' &= x - x_0 \\ v' &= v \end{aligned}\right\}$$

Invariant variables are t and v
$\Rightarrow \; v = v(t)$

group invariant solution

$$v_t = 0 \quad \Rightarrow \quad v = \text{const.}$$

iv) combination of time and space translations

$t' = t - t_0$

$x' = x - x_0$

$v' = v$

Invariant variables.

$\xi = x_0 t - t_0 x = x_0 (x - ct)$

$c = t_0 / x_0$

$v(\xi)$

omitting x_0 (take it 1)

$v(\xi)$, $\xi = x - ct$

$$v_t - 6vv_x + 6v_{xxx} = 0$$

$$v_t = -c\,v_\xi\,,\quad v_x = v_\xi\,,\quad v_{xx} = v_{\xi\xi}\,,\quad v_{xxx} = v_{\xi\xi\xi}$$

$$\Rightarrow\ -c\,v' - 6vv' + v''' = 0$$

can be integrated once

$$v'' - 2v^3 - cv + k = 0\,,\quad k = \text{constant}$$

v) Galileo transformation

$$u' = u - \lambda$$
$$x' = x + 6\lambda t$$
$$t' = t$$

invariant variables are

$$u + \frac{x}{6t} = u' + \frac{x'}{6t'}$$

and t

$$\zeta(t) = u + \frac{x}{6t}$$

$$u = \zeta(t) - \frac{x}{6t}$$

$$v_t = \frac{x}{6t^2} + \zeta'$$

$$v_x = \frac{1}{6t}$$

$$v_{xx} = 0$$

$$\frac{x}{6t^2} + \zeta_t - 6\left(\zeta + \frac{x}{6t}\right)\left(\frac{1}{6t}\right) = 0$$

$$\frac{x}{6t^2} + \zeta_t + \frac{1}{t}\zeta - \frac{x}{6t^2}$$

$$\zeta_t + \frac{1}{t}\zeta = 0$$

$$(t\zeta)_t = 0 \Rightarrow$$

$$\zeta = \frac{k}{t}$$

$$v = \frac{k}{t} - \frac{x}{6t}$$

vi) A combination of the Galileo transformation and time translation

$$v' = v - \lambda$$

$$x' = x + 6\lambda t \mp 3\lambda t_0$$

$$t' = t - t_0$$

$$v_t - 6 v v_x + v_{xxx} = v'_{t'} + v'_{x'}(6\lambda) - 6(v' + \lambda) v'_{x'} + v'_{x'x'x'}$$

$$= v'_{t'} - 6 v' v_{x'} + v'_{x'x'x'}$$

Invariant quantities

$$v + ct = v' + ct' = v - \lambda + ct' - ct_0$$

$$\lambda = -ct_0$$

$$\zeta_1 = v + ct$$

$$x - 3ct^2 = x' - 3ct'^2 = x + 6\lambda t \mp 3\lambda t_0 - 3c(t^2 - 2tt_0 + t_0^2)$$

$$= x - 3ct^2 + 6(\lambda + ct_0)t \mp 3\lambda t_0 - 3ct_0^2$$

$$= x - 3ct^2$$

$$\xi = x - 3ct^2$$

$$v(x,t) = \zeta_1(\xi) - ct$$

$$V_t = \zeta_1'(-6ct) - c$$

$$V_x = \zeta_1', \quad V_{xx} = \zeta_1'', \quad V_{xxx} = \zeta'''$$

$$\Rightarrow \quad V_t - 6VV_x + V_{xxx} = 0$$

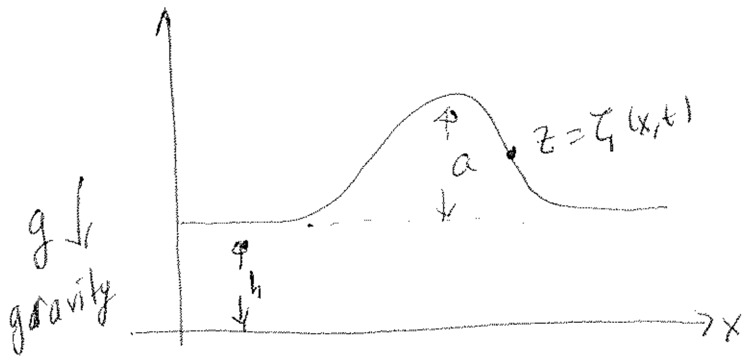
$$-6ct\zeta_1' - c - 6(\zeta_1 - ct)\zeta' + \zeta''' = 0$$

$$\zeta_1''' - 6\zeta\zeta' - c = 0$$

$$\zeta_1'' - 3\zeta_1^2 - c\xi + \alpha = 0$$

First Painleve' transcendent.

Discovery of Solitary Waves:



$\zeta(x,t)$ satisfies the KdV equation

$$\zeta_t = \frac{3}{2}\left(\frac{g}{h}\right)^{1/2}\left[\zeta\zeta_x + \frac{2}{3}\varepsilon\,\zeta_x + \frac{1}{3}\sigma\,\zeta_{xxx}\right] \quad (*)$$

σ is related to the surface tension

This equation can be transformed to

$$u_t - 6uu_x + u_{xxx} = 0$$

(The standard form of the KdV equation)

Eq $(*)$ has solitary wave solution. (Travelling waves)

Let $\zeta = \zeta(x - vt)$, v is the speed of the wave

using this in $(*)$ we get

$$-v\zeta' = \frac{3}{2}\left(\frac{g}{h}\right)^{1/2}\left[\zeta\zeta' + \frac{2}{3}\varepsilon\zeta' + \frac{1}{3}\sigma\zeta'''\right]$$

$$-v\zeta' = \frac{3}{2}(g/h)^{1/2}\cdot\frac{2}{3}\varepsilon\zeta' + \frac{3}{2}(g/h)^{1/2}\left[\zeta\zeta' + \frac{\sigma}{3}\zeta'''\right]$$

$$-\left[v + \varepsilon(g/h)^{1/2}\right]\zeta' = \frac{3}{2}(g/h)^{1/2}\left[\zeta\zeta' + \frac{\sigma}{3}\zeta'''\right]$$

integrating this equation we get

$$A - \left[v + \varepsilon(g/h)^{1/2}\right]\zeta = \frac{3}{2}(g/h)^{1/2}\left[\frac{1}{2}\zeta^2 + \frac{\sigma}{3}\zeta''\right]$$

A is an integration constant. Multiplying both sides by ζ' and integrating we get

$$B + A\zeta - \frac{1}{2}\left[v + \varepsilon(g/h)^{1/2}\right]\zeta^2 = \frac{3}{2}(g/h)^{1/2}\left[\frac{1}{6}\zeta^3 + \frac{\sigma}{6}\sigma'^2\right]$$

$$\frac{1}{4}\left[\zeta^3 + \sigma\zeta'^2\right] = \left(\frac{h}{g}\right)^{1/2}(B + A\zeta) - \frac{1}{2}\left[v\left(\frac{h}{g}\right)^{1/2} + \varepsilon\right]\zeta^2$$

$$\sigma\zeta'^2 + \zeta^3 + 2\left[v(h/g)^{1/2} + \varepsilon\right]\zeta^2 - 4(h/g)^{1/2}(B + A\zeta) = 0$$

Letting $A = B = 0$ we get

$$\sigma\zeta'^2 + \zeta^3 + 2\left[v(h/g)^{1/2} + \varepsilon\right]\zeta^2 = 0$$

Let

$$\zeta_1 = \alpha \operatorname{sech}^2(\beta \xi), \quad \xi = x - vt$$

$$\zeta_1' = \alpha\beta \frac{-2\sinh(\beta\xi)}{\cosh^3\beta\xi}$$

$$4\sigma\alpha^2\beta^2 \frac{\sinh^2\beta\xi}{\cosh^6\beta\xi} + \alpha^3 \frac{1}{\cosh^6\beta\xi}$$

$$+ 2\left[v (h/g)^{1/2} + \varepsilon\right] \frac{\alpha^2}{\cosh^4\beta\xi} = 0$$

$$4\sigma\alpha^2\beta^2 \sinh^2\beta\xi + 2\alpha^2\left[v(h/g)^{1/2} + \varepsilon\right]\cosh^2\beta\xi$$
$$+ \alpha^3 = 0$$

$$4\sigma\alpha^2\beta^2 = -2\alpha^2\left[v(h/g)^{1/2} + \varepsilon\right]$$

$$-4\sigma\alpha^2\beta^2 + \alpha^3 = 0 \Rightarrow \alpha = 4\sigma\beta^2$$
$$\alpha \neq 0$$

$$2\sigma\beta^2 = -\left[v(h/g)^{1/2} + \varepsilon\right]$$

$$\alpha = -2\left[v(h/g)^{1/2} + \varepsilon\right]$$

$$\Rightarrow \quad v = -(\alpha/2 + \varepsilon)\sqrt{g/h} = -\left(\frac{\alpha}{2h} + \frac{\varepsilon}{h}\right)\sqrt{gh}$$

$$\alpha = 4\sigma\beta^2.$$

$$\zeta_1(x,t) = 4\sigma\beta^2 \operatorname{sech}^2(\beta\xi)$$

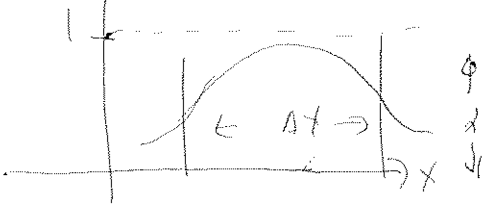
$$= \alpha \operatorname{sech}^2\left[\sqrt{\frac{\alpha}{4\sigma}}\,(x - vt)\right]$$

$$v = -\left(\frac{\alpha}{2h} + \frac{\varepsilon}{h}\right)\sqrt{gh}$$

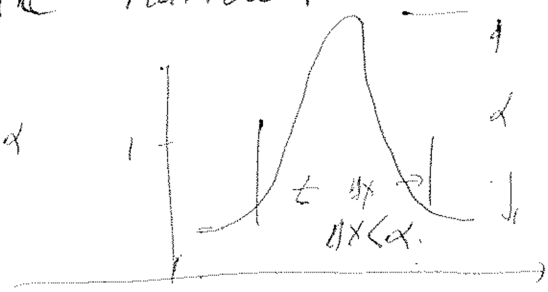
1) speed of the wave is proportional to the amplitude α
Here, since ε is small we can ignore it

Taller waves move faster.
i.e if α is large v is large

2) The width of the wave
$\Delta x \sim \alpha^{1/2}$
taller waves are narrower.



$\Delta x \not\propto \alpha$



This solution is called the travelling wave solution of the KdV equation. For the standard form of the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0$$

let $\xi = x - vt$, $u(x,t) = f(\xi)$

$$-cf' - 6ff' + f''' = 0$$

$$\Rightarrow \quad -cf - 3f^2 + f'' = A$$

$$-\frac{c}{2}f^2 - f^3 + \frac{1}{2}f'^2 = Af + B$$

$$\frac{1}{2}f'^2 = \frac{c}{2}f^2 + f^3 + Af + B \quad , \quad A, B \text{ are const.}$$

for solitary waves $A = B = 0$ (rapidly decaying) $f, f', f'' \to 0$ as $\xi \to \pm\infty$

$$\frac{1}{2}f'^2 = \frac{c}{2}f^2 + f^3$$

$$\Rightarrow \quad f(x,t) = -\frac{1}{2}v \operatorname{sech}^2\left[\frac{1}{2}v^{1/2}(x - vt)\right]$$

General waves of permanent form

After a symmetry reduction a wave equation reduces to an ODE of the form

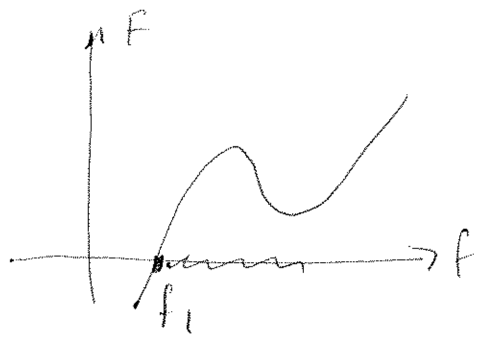
$F(f) = \frac{1}{2} f'^2$ where

$F(f) = f^3 + \frac{1}{2} c f^2 + Af + B$

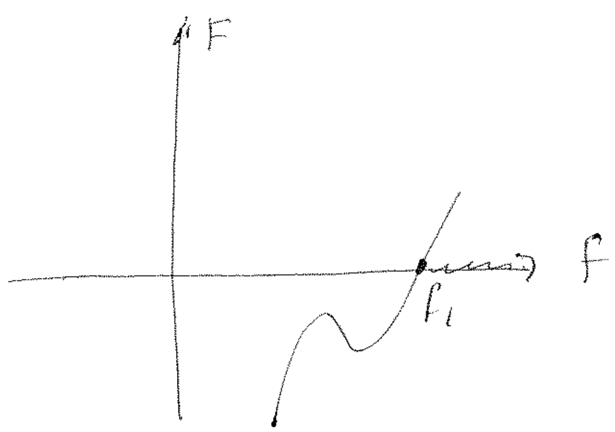
if $f', f'' \to 0$ as $\xi \to \pm\infty \Rightarrow A = B = 0$ (rapidly decaying solutions). If there are not satisfied A and B are not equal to zero in general.

There may be the following cases

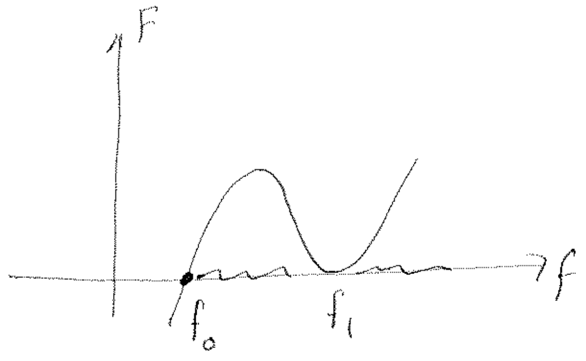
a) One root of $F(f)$



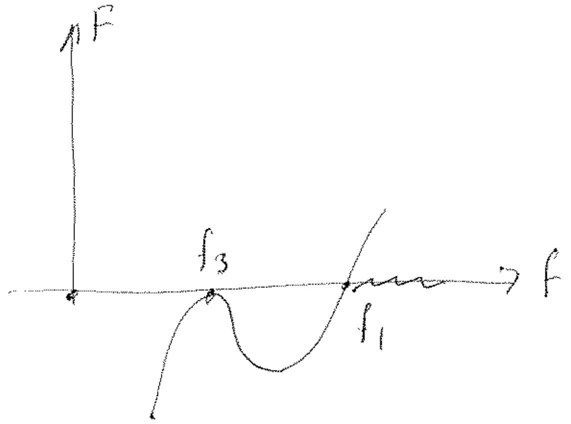
(b)



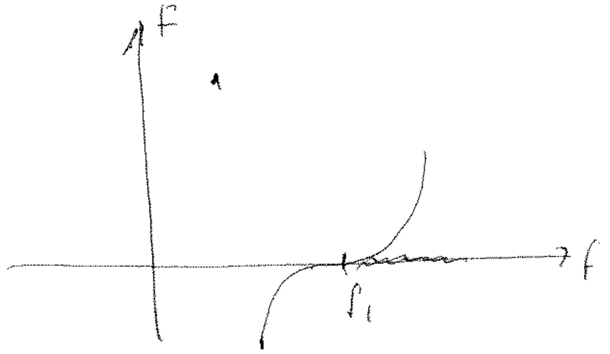
c) Two roots of F(f)



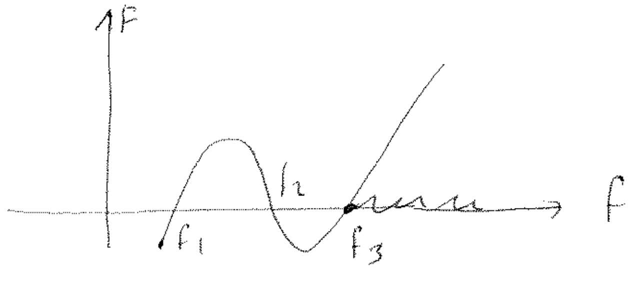
d) Two roots of F(f)



e) Triple root : All roots are equal.



f) Different root



In the neighborhood of these zeros

a) One simple zero: $F(f) = (f - f_1)\, G(f)$

$\frac{1}{2} f'^2 = F(f)$, $\quad F(f) = F(f_1) + F'(f_1)(f-f_1)$

$O((f-f_1)^2)$

$= F'(f_1)(f-f_1) + O(f-f_1)^2$

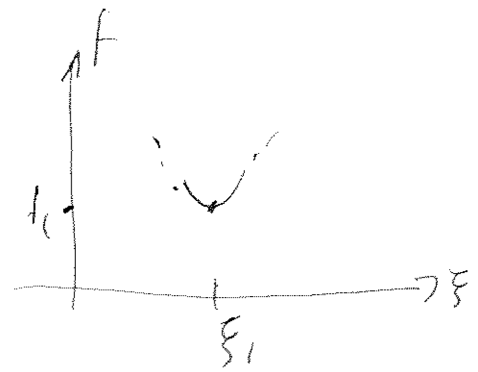
$\Rightarrow f'^2 = 2(f-f_1)F'(f_1) + O((f-f_1)^2)$

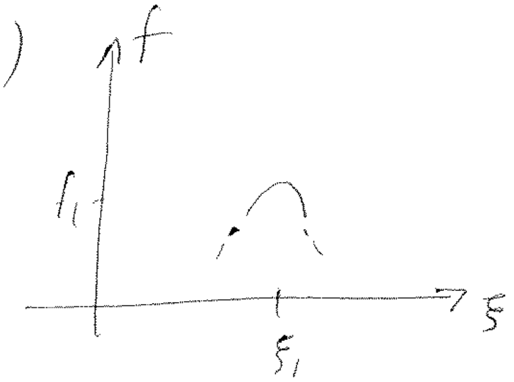
$\Rightarrow f(\xi) = f_1 + \frac{1}{2}(\xi - \xi_1)^2 F'(f_1) + O((\xi-\xi_1)^2)$

In the neighborhood of ξ_1 we have

i) a ~~maximum~~ minimum (concave up) if $F'(f_1) > 0$

ii) a maximum (concave down) if $F'(f_1) < 0$

i) 

ii) 

b) If f_1 is a double zero

$$F(f) = F(f_1) + F'(f_1)(f-f_1) + \frac{1}{2}(f-f_1)^2 F''(f_1)$$

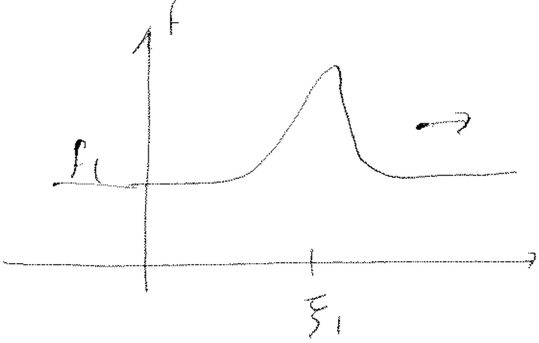
$$O[(f-f_1)^3]$$

$$F(f_1) = 0, \quad F'(f_1) = 0$$

$$F(f) = \frac{1}{2} f'^2 = \frac{1}{2}(f-f_1)^2 F''(f_1)$$

Valid only for $F''(f_1) > 0$

$$f(\xi) = f_1 + \alpha e^{\pm\sqrt{F''(f_1)}\,\xi} \quad \text{as } \xi \to \pm\infty$$



Thus $f \to f_1$ as $\xi \to \mp\infty$ and the solution can therefore have only one peak and the wave must extend from $-\infty$ to ∞

$$f(\xi) - f_1 = \alpha e^{-\sqrt{F''}\,\xi} \quad \text{as } \xi > 0$$

$$= \alpha e^{\sqrt{F''}\,\xi} \quad \text{as } \xi < 0$$

c) If f_1 is a triple zero then there is only one possibility namely

$$F(f) = A_0 (f - f_1)^3$$

$$f_1 = -c/6 \; , \; A = 3/\left(\frac{c}{6}\right)^2 \; , \; B = 1/\left(\frac{c}{6}\right)^3$$

$$f'^2 = 2A_0 (f - f_1)^3$$

$$f(\xi) = -\frac{c}{6} + \frac{2}{(\xi - \beta)^2}$$

$$f' = -\frac{4}{(\xi - \beta)^3}$$

$$\frac{16}{(\xi - \beta)^6} = 2A_0 \frac{8}{(\xi - \beta)^6} \Rightarrow A_0 = 1.$$

β arbitrary

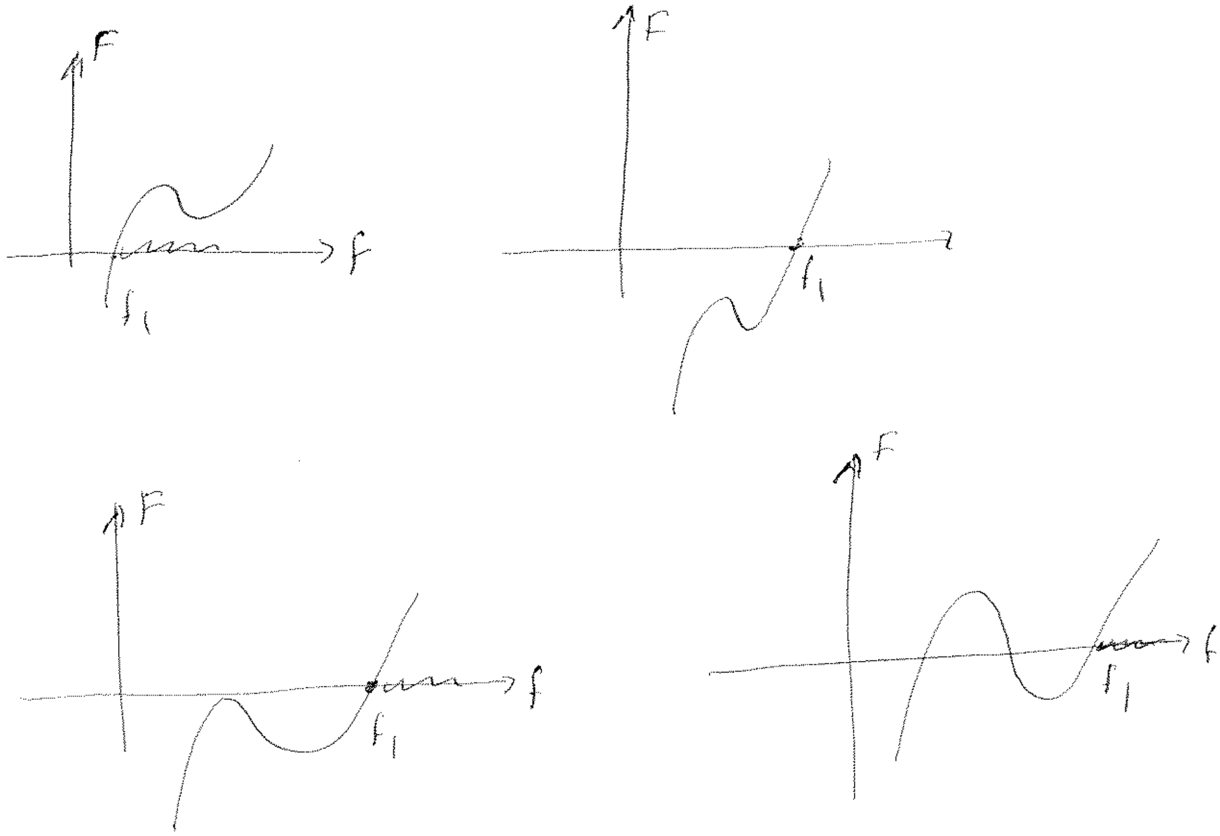
This solution is singular at $\xi = \beta$.

Thus ignoring this last case which is singular

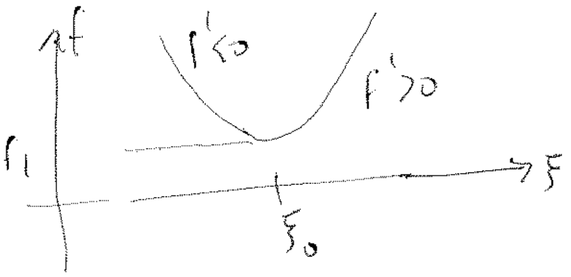
f will either change sign across $f = f_1$

or $f' \to 0$ as $\xi \to \pm \infty$

A) Consider now the cases

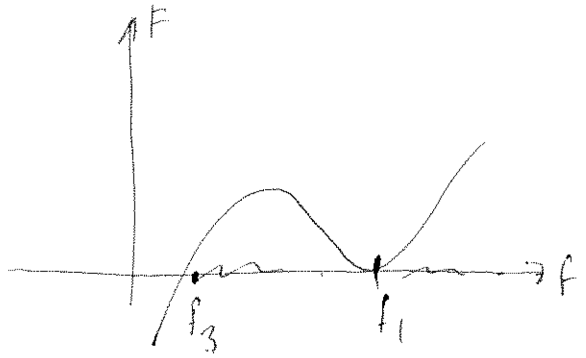


If at some point $\xi = \xi_0$ on the solution the slope is such that $f' > 0$, then $F > 0$ for all $\xi > \xi_0$ and $f \to \infty$ as $\xi \to \infty$



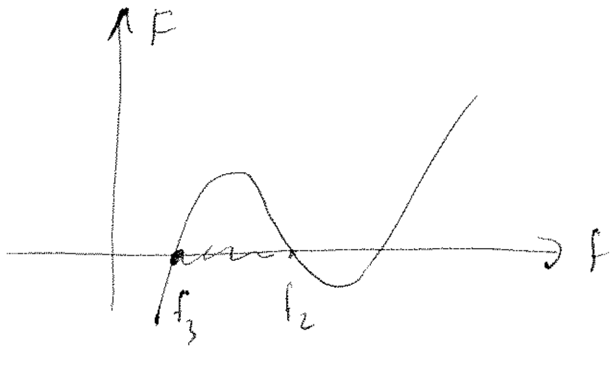
If however $f'(\xi) < 0$ then f will decrease until it reaches to f_1 this is a simple zero and so f' changes sign and once again $f \to \infty$ as $\xi \to \infty$. Hence for these four cases there is no bounded solution

B) For the case

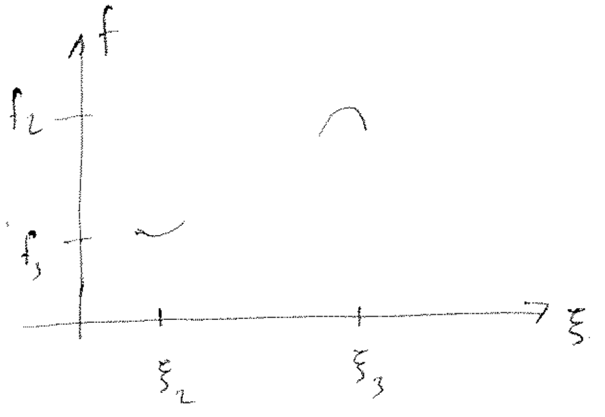


F has a simple zero at f_3 and a double zero at f_1. The solution has a minimum at f_3 ($F' > 0$) and attains $f = f_1$ as $\xi \to \pm\infty$. This is of course the "solitary wave" solution with amplitude $f_3 - f_1$

C) Finally we are left with the ~~right~~ left hand side of the case



We have simple zero at both f_3 and f_2. In fact we have a local minimum at f_3 and a local maximum at f_2. Hence f' changes sign at these pts. Since the behaviour near them is algebraic, consequtive pts $f = f_2$ and $f = f_3$ will be a finite distance apart.



If f and f' is given in between ξ_2 and ξ_3 which fixes the curve in between f_2 and f_3. The solution oscillates between f_2 and f_3

Elliptic functions (Jacobi Elliptic Functions)

When the function F is a polynomial of degree 3 then the solutions of the equation

$$\frac{1}{2} f'^2 = F(f)$$

are given in terms of the Jacobi elliptic functions.

Elliptic Functions:

Define the integral

$$v = \int_0^\phi \frac{d\theta}{\sqrt{1 - m\sin^2\theta}} \qquad 0 \leq m \leq 1 \qquad (*)$$

To get an idea of such integrals consider the following integral

$$w = \int_0^\psi \frac{dt}{\sqrt{1-t^2}} = \int_0^\psi d\theta = \sin^{-1}\psi$$

$$t = \sin\theta$$

$$w = \sin^{-1}\psi$$

similarly the integral (*) is the inverse of a function

$$\vartheta = sn^{-1}(\sin\phi)$$

or $\quad sn\,\vartheta = \sin\phi$

Define

$$cn\,\vartheta = \cos\phi$$

Then $\quad sn^2\vartheta + cn^2\vartheta = \sin^2\phi + \cos^2\phi = 1.$

Special cases:

$m=0 \qquad \vartheta = \int_0^{\phi} d\theta = \phi$

$$\Rightarrow \sin\vartheta = sn\,\phi = \sin\phi$$

sn function reduces to sin function

$m=1 \qquad \vartheta = \int_0^{\phi} \frac{d\theta}{\cos\theta} = \text{sech}^{-1}(\cos\phi)$

$$\text{sech}\,\vartheta = \cos\phi$$

$$\Rightarrow cn\,\vartheta = \text{sech}\,\vartheta$$

$$sn\,\vartheta = \tanh\vartheta$$

since $sn\,\theta = \sin\phi$

periodic function

$\phi \to \phi + 2\pi$

$\theta \to \theta + K$

$sn(\phi + K) = \sin\phi$

$cn(\phi + K) = cn\,\phi$

where

$$K = \int_0^{2\pi} \frac{d\theta}{(1 - m\sin^2\theta)^{1/2}}, \quad K(0) = 2\pi$$

Derivative of sn function and cn function

1) $\frac{d}{dv} cn\,v = \frac{d}{d\phi} cn\,v \frac{d\phi}{dv}$ $\quad cn\,v = \cos\phi$

$= -\sin\phi \cdot (1 - m\sin^2\phi)$

$\left(\frac{d}{dv} cn\,v\right)^2 = (1 - cn^2 v)(1 - m + m\,cn^2 v)$

let $cn\,v = y$

$y'^2 = (1 - y^2)(1 - m + m y^2)$

RHS in Fourth degree polynomial.

2) let $y = sn^2 v$,

$$y' = \cos\phi\ (1 - m\sin^2\phi)$$

$$y'^2 = (1-y^2)(1-my^2)$$

3) $$y = cn^2 v$$

$$y' = 2(cn\,v)'\ cn\,v$$

$$(cn\,v)' = -\sin\phi\ (1 - m\sin^2\phi)^{1/2}$$

$$y'^2 = 4\,(1-y)\,(1 - m\,(1-y))\,y$$

$$= 4\,y\,(1-y)(1 - m + my)$$

RHS a third order polynomial

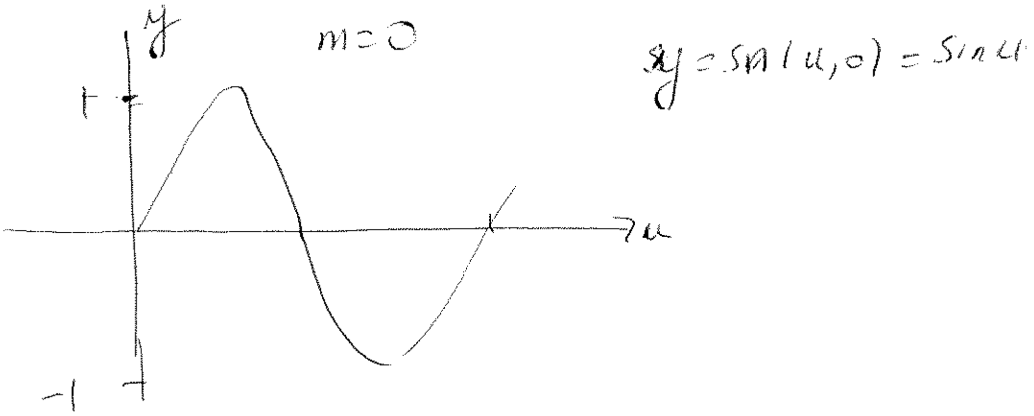
4) $$y = sn^2 v$$

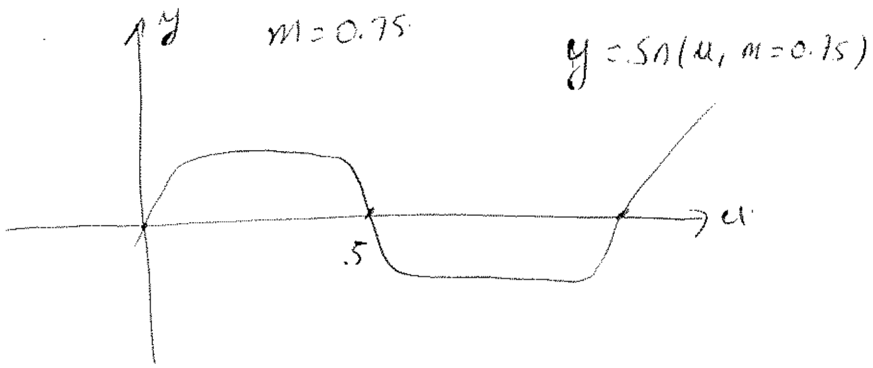
$$y' = 2(sn\,v)'\ sn\,v \quad = 2\cos\phi\ (1 - m\sin^2\phi) \cdot sn\,v$$

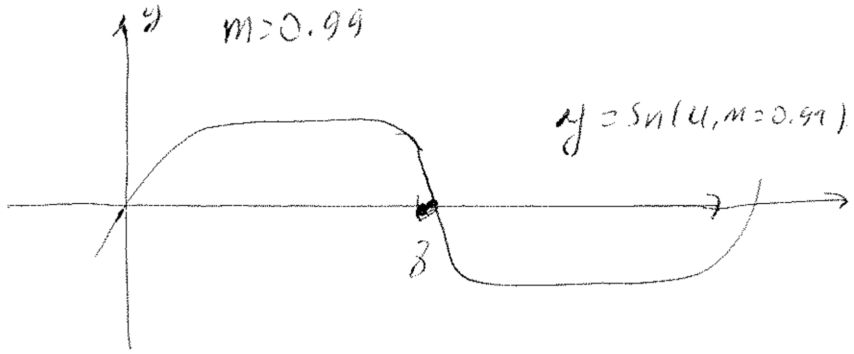
$$y'^2 = 4(1-y)\,y\,(1-my)$$

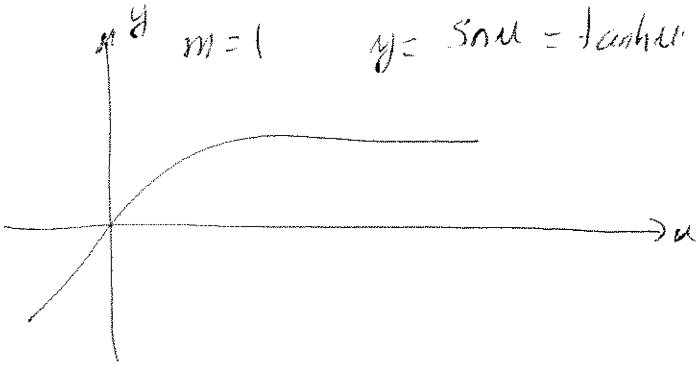
RHS is a third degree polynomial.

Some examples of Jacobi elliptic functions









In our general analysis (travelling wave solution) we have

$$\frac{1}{2}\zeta_1'^2 = 2(\zeta_1-\zeta_{1})(\zeta_1-\zeta_2)(\zeta_1-\zeta_3) \quad , \quad \zeta_1' = \frac{d\zeta_1}{d\xi}$$

where ζ_1, ζ_2, and ζ_3 are the roots of the third degree polynomial $F(\zeta_1)$. Now we assume that

$$\zeta_1(\xi) = \alpha + \beta\, y(a\xi)$$

where y is one of the Jacobi elliptic functions. Here α, β, and a are constants to be determined.

$$\zeta_1' = a\beta\, y_\rho(\rho) \quad , \quad \rho = a\xi \quad , \quad \zeta_1' = \frac{d\zeta_1}{d\xi}$$

$$a^2\beta^2(y_\rho)^2 = 2(\alpha-\zeta_1+\beta y)(\alpha-\zeta_2+\beta y)(\alpha-\zeta_3+\beta y)$$

Choose

(i) $\alpha = \zeta_2$

$$a^2\beta^2(y_\rho)^2 = 2\beta\, y(\zeta_2-\zeta_1+\beta y)(\zeta_2-\zeta_3+\beta y)$$

or

$$a^2 \beta \left(y_\xi\right)^2 = 2(\zeta_{12}-\zeta_{13})(\zeta_{12}-\zeta_{11})\, y \left(1+\frac{\beta}{\zeta_{12}-\zeta_{13}}\, y\right)\left(1+\frac{\beta}{\zeta_{12}-\zeta_{11}}\, y\right)$$

let $y = \mathrm{cn}(\xi)$

$$\left(y_\xi\right)^2 = 4y(1-y)(1-m+my) = 4(1-m)\, y(1-y)\left(1+\frac{m}{1-m}\, y\right)$$

$\Rightarrow$ (ii) $2a^2\beta(1-m) = (\zeta_{12}-\zeta_{13})(\zeta_{12}-\zeta_{11})$

(iii) $\beta = -\zeta_{12}+\zeta_{13}.$

iv) $\dfrac{\beta}{\zeta_{12}-\zeta_{11}} = -\dfrac{m}{m-1}$

$\Rightarrow$ $\alpha = \zeta_{12}$, $\beta = \zeta_{13}-\zeta_{12}$ $\quad 2a^2(1-m) = -\zeta_{12}+\zeta_{11}$

$$-\frac{m}{m-1} = \frac{\zeta_{13}-\zeta_{12}}{\zeta_{12}-\zeta_{11}} \Rightarrow -\frac{m-1}{m} = \frac{\zeta_{12}-\zeta_{11}}{\zeta_{13}-\zeta_{12}}$$

$$-1+\frac{1}{m} = \frac{\zeta_{12}-\zeta_{11}}{\zeta_{13}-\zeta_{12}} \Rightarrow \frac{1}{m} = 1+\frac{\zeta_{12}-\zeta_{11}}{\zeta_{13}-\zeta_{12}} = \frac{\zeta_{13}-\zeta_{11}}{\zeta_{13}-\zeta_{12}}$$

$$m = \frac{\zeta_{13}-\zeta_{12}}{\zeta_{13}-\zeta_{11}}$$

$$1 - m = \frac{\zeta_{12} - \zeta_{11}}{\zeta_{13} - \zeta_{11}} \Rightarrow$$

$$2a^2 \left(\frac{\zeta_{12} - \zeta_{11}}{\zeta_{13} - \zeta_{11}} \right) = -(\zeta_{12} - \zeta_{11})$$

$$\Rightarrow 2a^2 = \zeta_{11} - \zeta_{13}.$$

$$a = \pm \frac{1}{\sqrt{2}} \sqrt{\zeta_{11} - \zeta_{13}}$$

all parameters are determined in terms of the roots of F

$$\zeta_1(\xi) = \zeta_{12} - (\zeta_{12} - \zeta_{13}) \, cn^2 \left[\sqrt{\frac{\zeta_1 - \zeta_3}{2}} (\xi - \xi_0) \right]$$

$$m = \frac{\zeta_{13} - \zeta_{12}}{\zeta_{13} - \zeta_{11}}$$

<u>Special cases</u>

a) $\zeta_{13} \to \zeta_{12}$ (double root) $cn \to \cos$. oscillating soluh

b) $\zeta_{11} \to \zeta_{12}$ (n double roo) $cn \to sech$.

Since

$$F(\zeta_1) = 2\left(\zeta_1^3 + \frac{1}{2}c\zeta_1^2 + A\zeta_1 + B\right)$$

$$= 2(\zeta_1 - \zeta_{11})(\zeta_1 - \zeta_{12})(\zeta_1 - \zeta_{13})$$

we assume $\zeta_{13} < \zeta_{12} < \zeta_{11}$

and we had the speed of the wave as

$$c = \zeta_{11} + \zeta_{12} + \zeta_{13}$$

Problems

1) Determine the values of m and n so that $u(x,t) = t^m f(xt^n)$ is a solution of the Burgers equation

$$u_t + uu_x = u_{xx}$$

and write down the equation satisfied by f. Hence obtain the solution for which $f \to 0$ as $x \to \infty$, and $f(0) = -2/\pi^{1/2}$

2) Show that the modified KdV equation

$$u_t + 6u^2 u_x + u_{xxx} = 0$$

is invariant under the transformation
$x \to \lambda x, \quad t \to \lambda^3 t, \quad u \to \lambda^{-1} u \quad (\lambda \neq 0).$
Hence introduce $u(x,t) = t^{-1/3} f(xt^{-1/3})$ and show that f satisfies

$$f'' - \frac{1}{3}\xi f + 2f^3 = 0, \quad \xi = x t^{-1/3}$$

provided $f \to 0$ sufficiently rapidly at infinity

3) Show that the nonlinear Schrodinger equation

$$i u_t + u_{xx} + \nu u |u|^2 = 0$$

where ν is a real constant, is invariant under each of the group transformations

i) $t \to t + \lambda, \quad x \to x, \quad u \to u$

ii) $t \to t, \quad x \to x + \lambda, \quad u \to u$

iii) $t \to \lambda^2 t, \quad x \to \lambda x, \quad u \to \lambda^{-1} u \quad (\lambda \neq 0)$

Hence use property (iii) to suggest a similarity solution of the form

$$u(x,t) = t^m f(x t^n)$$

for suitable m and n and obtain the equation for f

4) Find the travelling wave solution of the equations

a) A generalized KdV eqn

$$u_t + (n+1)(n+2) u^n u_x + u_{xxx} = 0$$

$n = 1, 2, \ldots$ with $u, u_x, u_{xx} \to 0$ as $|x| \to \infty$

b) An elastic-medium equation

$$u_{tt} = u_{xx} + u_x u_{xx} + u_{xxxx}$$

with $u_x, u_{xx}, u_{xxx} \to 0$ as $|x| \to \infty$

5) In (4a) with the sign of the nonlinear term as negative

$$u_t - (n+1)(n+2)u^n u_x + u_{xxx} = 0$$

with $n = 1, 2, \ldots$ and

$u, u_x, u_{xx} \to 0$ as $|x| \to \infty$.

Show that solitary wave solutions exists only if n is odd integer.

Appendix A

Linear and Nonlinear Waves

Linear Waves:

Let us consider the simplest wave equation

$$u_t + c\, u_x = 0 \quad , \quad x \in \mathbb{R},\ t > 0$$

where c is a constant, denotes the speed of the wave. The general solution of this equation is a right-travelling wave

$$u(x,t) = f(x - ct)$$

where f is an arbitrary function. If we give the initial condition

$$u(x,0) = \phi(x) \quad , \quad x \in \mathbb{R}$$

then we find that $f(x) = \phi(x)$. Hence the solution is

$$u(x,t) = \phi(x - ct).$$

Characteristic lines: The straight lines $x - ct =$ constant play an important role. Along these lines the initial values are propagated with constant value (not changing on the characteristic lines $x - ct =$ constant

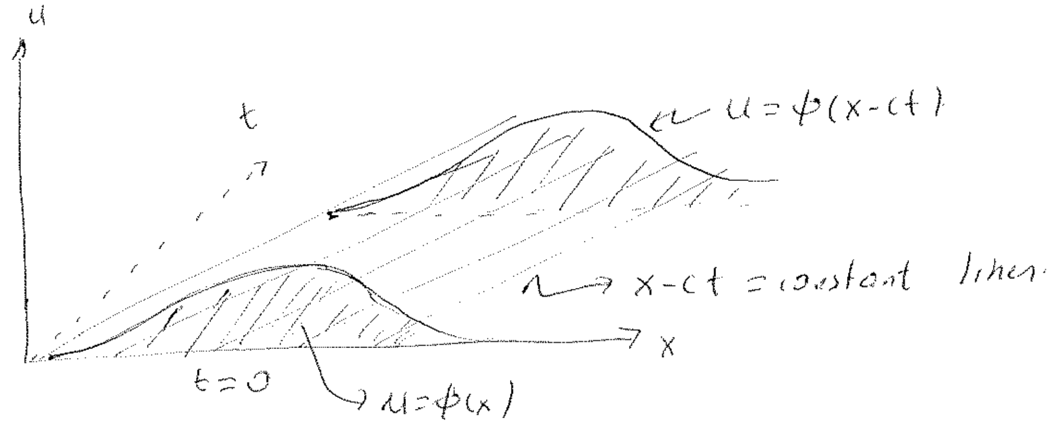
Along these lines $u(x(t),t)$

$$\frac{du}{dt} = u_x \frac{dx}{dt} + u_t$$

$\frac{dx}{dt} = c$ speed of the wave.

Hence along the curve $x - ct = \text{const}$

$\frac{du}{dt} = 0 \Rightarrow u = \text{constant}$ on the characteristic curve



Family of straight lines $x - ct = k$ (constant) is called the family characteristic curves for this problem $u_t + cu_x = 0$

Now consider a more complicated problem

$$u_t + c(x,t)\, u_x = 0 \quad , \; x \in \mathbb{R}, \; t > 0$$

$$u(x,0) = \phi(x) \quad , \; x \in \mathbb{R}$$

where $\phi(x)$ and $c(x,t)$ are given functions.

To find the family of characteristic curves for this problem can be found by using the Lagrange method

$$\frac{dt}{1} \pm \frac{dx}{c} = \frac{du}{0}$$

i) $v_1 = u = c_1$.

ii) $v_2 \Rightarrow dx = c(x,t)dt$
solution of this define the characteristic curves. ∴ The speed of the wave

$$\frac{dx}{dt} = c(x,t)$$

Hence the speed of the wave depends on x and t.

Again u is constant along the characteristic curve: On this curve $u = u(x(t), t)$

$$\frac{du}{dt} = u_x \frac{dx}{dt} + u_t = u_t + c(x,t)u_x = 0$$

Hence u = constant on the family of characteristic curves

As an example consider the problem: $(c(x,t) = 2t)$

$$u_t + 2t\, u_x = 0 \qquad x \in \mathbb{R},\ t > 0$$

$$u(x,0) = e^{-x^2}, \qquad x \in \mathbb{R}.$$

Characteristic curves

$$\frac{dx}{dt} = 2t \Rightarrow x - t^2 = k \text{ constant.}$$

If at $t = 0$, $x = \xi \Rightarrow x - t^2 = \xi$

or $x = t^2 + \xi$ (characteristic lines)

SOLUTION OF THE PROBLEM

$$\frac{dt}{1} = \frac{dx}{2t} = \frac{du}{0} \Rightarrow u(x,t) = f(x - t^2)$$

since at $t = 0$ $u(x,0) = e^{-x^2}$

hence $f(x) = e^{-x^2} \Rightarrow$

$$u(x,t) = e^{-(x-t^2)^2}$$

Nonlinear Waves:

Let us consider similar type of problems

$$u_t + c(u) u_x = 0 , \quad x \in \mathbb{R} , \ t > 0$$

where $c'(u)$ (an increasing function). With the initial condition

$$u(x,0) = \phi(x) , \quad x \in \mathbb{R}$$

where ϕ is a given function.

The characteristic curves are given by

$$\frac{dx}{dt} = c(x,t) \qquad (*)$$

Again u remains constant on these curves. i.e $u = u(x(t), t)$

$$\frac{du}{dt} = u_x \frac{dx}{dt} + u_t = u_t + c(u) u_x = 0$$

$\Rightarrow u = \text{constant}$ on the curve $x = x(t)$ as the solution of the eqn. $(*)$. These are the characteristic curves. Since

$$\frac{d^2 x}{dt^2} = \frac{d}{dt} c(u) = c'(u) \frac{du}{dt} = 0 \Rightarrow$$

$$x = c(u) t + \xi$$

curves start at $x = \xi$ when $t = 0$. Hence the characteristic curves are

$$x = c(u)t + \xi \qquad (*)$$

and the speed of the wave is $c(u)$
Using the Lagrange method the solution of the problem is

$$u(x,t) = f(\xi) = f(x - c(u)t)$$

where f is an arbitrary function. Since

$$u(x,0) = f(x) = \phi(x)$$

Then the solution of this problem is

$$u(x,t) = \phi(\xi) = \phi(x - c(u)t)$$

$$= \phi(x - c(\phi(\xi))t)$$

Hence, if the initial condition

$$u(x,0) = \phi(x)$$

is given ($\phi(x)$ is a known function)
Then the general solution for $t > 0$ is

$$u(x,t) = \phi(\xi) = \phi(x - c(\phi(\xi))t) \qquad (**)$$

where ξ is given in $(*)$ above

Example: consider the following initial value problem

$$u_t + u u_x = 0, \quad x \in \mathbb{R}, \quad t > 0$$

$$\phi(x) = \begin{cases} 2, & x < 0 \\ 2-x & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

Initial wave profile

In this example $c(u) = u$. Hence in each interval $\xi = x - u(x,t)\,t$

i) $x < 0$ $\qquad \phi(x) = 2$

using $(**)$ we get

$\xi = x - 2t$, $\quad u(x,t) = 2$ for $\xi < 0$

or $x < 2t$

ii) $0 \leq x \leq 1$ $\qquad \phi(x) = 2 - x$

using $(**)$ we get

$u(x,t) = 2 - \xi$, $\quad$ but

$$x - c(\phi(\xi))\, t = \xi$$

$$\Rightarrow \quad x - \phi(\xi)\, t = \xi$$

$$\Rightarrow \quad x - (2 - \xi)\, t = \xi$$

$$\Rightarrow \quad x - 2t = \xi(1-t) \quad \Rightarrow \quad \xi = \frac{x - 2t}{1-t}$$

There is a singularity at $t = 1$. Hence what we are doing is valid up to $t = 1$ $(t < 1)$

The general solution is then

$$u(x,t) = 2 - \frac{x-2t}{1-t} = \frac{2 - 2t - x + 2t}{1-t}$$

$$u(x,t) = \frac{2-x}{1-t} \qquad (t < 1)$$

and $\quad 0 \leq \xi \leq 1 \quad \Rightarrow \quad 0 \leq x - 2t \leq 1$

$(t < 1)$

$$2t \leq x \leq t + 1$$

iii) $\quad x > 1 \qquad \phi(x) = 1$

using (**) we get

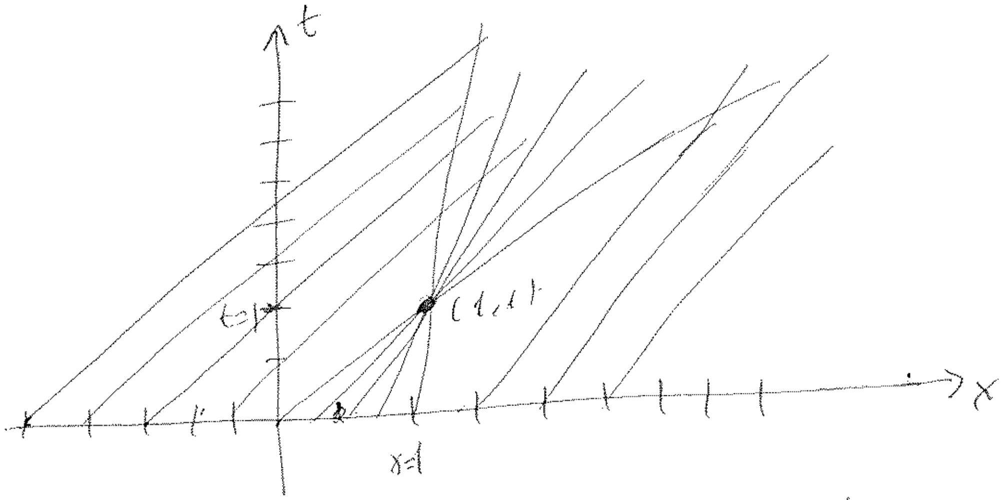
$$u(x,t) = 1, \qquad x - t = \xi$$

hence $\quad \xi > 1 \Rightarrow x > t + 1$

~~whole range of x is in~~

∴ x

We observe that the solution can not exist ~~beyond~~ for $t > 1$, since the characteristic lines cross beyond that time and they carry different constant values of u.



Characteristic diagram of the example

At $t = 1$ breaking of the wave occurs, which is the first instant when the solution becomes multiple valued as we have discussed earlier.

In general the initial value problem

$$u_t + c(u) u_x = 0, \quad x \in \mathbb{R}, \quad t > 0$$

with $c'(u) > 0$ with initial condition

$$u(x,0) = \phi(x), \quad x \in \mathbb{R}$$

may have solutions ~~....~~ only up to a finite time t_b, which is called the breaking time. Let us assume ~~that~~ in addition to $c'(u) > 0$ that the initial wave profile satisfies the condition

$$\phi(x) \geq 0, \quad \phi'(x) < 0$$

At the time ~~time~~ when breaking occurs the gradient u_x becomes infinite. To compute u_x we use

$$x = c(\phi(\xi))t + \xi$$

In page A6. Differentiating this implicitly wrt x we get

$$1 = c'(u)\,\phi'\,t\,\xi_x + \xi_x$$

$$\Rightarrow \quad \xi_x = \frac{1}{1 + c'(u)\phi' t}$$

Since (from page A6)

$$u(x,t) = \phi(\xi)$$

$$u_x = \phi' \xi_x = \frac{\phi'}{1 + c'(u)\phi' t}$$

The gradient catastrophe will occur at the minimum value of t which makes the denominator zero. Then

$$t_b = -\min_{\xi} \frac{1}{\phi'(\xi)c'(u)} \qquad t_b \geq 0$$

In the previous example: $c(u) = u$

$\phi(\xi) = 2 - \xi \Rightarrow \phi'(\xi)c'(u) = -1$.

$\Rightarrow \quad t_b = 1$ is the time when breaking occurs.

As a summary: we observed that the nonlinear pde

$$u_t + uu_x = 0 \quad , \quad c'(u) > 0 \qquad x \in \mathbb{R}$$

propagates the initial wave profile at a speed $c(u)$ depends on the value of the solution u at a given point. Since $c'(u) > 0$ large values of u are propagating faster than the small values and distortion of the wave profile occurs

Problems

1) Solve the following initial value problems on $t > 0$, $x \in \mathbb{R}$. Sketch the characteristic curves in each case

a) $u_t + 3u_x = 0, \quad u(x,0) = \sin x.$

b) $u_t + x u_x = 0, \quad u(x,0) = e^{-x}.$

c) $u_t - x^2 u_x = 0, \quad u(x,0) = x+1.$

2) Find the solution and sketch a characteristic diagram

$$u_t + c\, u_x = 0, \quad x \in \mathbb{R},\ t > 0,\ c > 0$$

$$u(x,0) = \begin{cases} 1-x^2, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

3) Consider the initial value problem

$$u_t + u u_x = 0, \quad x \in \mathbb{R},\ t > 0$$

$$u(x,0) = \begin{cases} 1-x^2, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

Sketch the characteristic diagram. At what time t_b does the wave break? Find a formula for the solution

4) Consider the initial value problem

$$u_t + uu_x = 0 \quad , \quad x \in \mathbb{R}\ , \ t > 0$$

$$u(x,0) = e^{-x^2} \quad , \quad x \in \mathbb{R}$$

Sketch the characteristic diagram and find the point (x_b, t_b) in spacetime where the the wave breaks.